

The S_3 flavor symmetry in 3-3-1 models

P. V. Dong,^{*} H. N. Long,[†] and C. H. Nam

Institute of Physics, VAST, 10 Dao Tan, Ba Dinh, Hanoi, Vietnam

V. V. Vien

*Department of Physics, Tay Nguyen University,
567 Le Duan, Buon Ma Thuot, Vietnam*

(Dated: November 29, 2011)

We propose two 3-3-1 models (with either neutral fermions or right-handed neutrinos) based on S_3 flavor symmetry responsible for fermion masses and mixings. The models can be distinguished upon the new charge embedding (\mathcal{L}) relevant to lepton number. The neutrino small masses can be given via a cooperation of type I and type II seesaw mechanisms. The latest data on neutrino oscillation can be fitted provided that the flavor symmetry is broken via two different directions $S_3 \rightarrow Z_2$ and $S_3 \rightarrow Z_3$ (or equivalently in the sequel $S_3 \rightarrow Z_2 \rightarrow \{\text{Identity}\}$), in which the second direction is due to a scalar triplet and another antisextet as small perturbation. In addition, breaking of either lepton parity in the model with neutral fermions or lepton number in the model with right-handed neutrinos must be happened due to the \mathcal{L} -violating scalar potential. The TeV seesaw scale can be naturally recognized in the former model. The degenerate masses of fermion pairs (μ, τ) , (c, t) and (s, b) are respectively separated due to the $S_3 \rightarrow Z_3$ breaking.

PACS numbers: 14.60.Pq, 14.60.St, 11.30.Hv, 12.60.-i

I. INTRODUCTION

The experiments of neutrino oscillations have indicated that the neutrinos have small masses and mixings [1], thus the standard model of fundamental particles and interactions must be extended. Among the proposals known today for explanation of the above problems, the seesaw mechanism [2] is perhaps the most popular and natural. In this scenario, the heavy right-handed neutrinos ν_R (or called neutral fermions N_R in some variants) are actually required so that the mechanism works. The presence of these particles can imply interesting cosmological consequences such as the baryon asymmetry via leptogenesis [3]. However, the mystery is that they have not been observed. What

^{*}Electronic address: pvdong@iop.vast.ac.vn

[†]Electronic address: hnlong@iop.vast.ac.vn

is the natural origin of them. There have been nice approaches in which they may be necessary constituents of the theory such as left-right symmetry [4] or $\text{SO}(10)$ grand unification [5].

An alternative is to extend the electroweak symmetry into $\text{SU}(3)_L \otimes \text{U}(1)_X$, in which to complete the fundamental representations of $\text{SU}(3)_L$ with the standard model doublets, the right-handed neutrinos or neutral fermions may be acquired. This proposal has nice features and been extensively studied over the last two decades, called 3-3-1 models [6–8]. Indeed, in the standard model as well as the theories mentioned the number of fermion families is left arbitrarily although from the experimental observations and fits we surely know that it is three. The reason possibly originates from the fact that the anomalies are canceled on every family, no interplay between families needed [9] (see also P. H. Frampton in [7]). In the standard model this cancelation is due to the chiral electroweak symmetry $\text{SU}(2)_L \otimes \text{U}(1)_Y$ with the relevant $\text{SU}(2)_L$ trace $\text{Tr}[\{T_a, T_b\}T_c] = 0$ for any fermion representation. The simplest extension to the $\text{SU}(3)_L \otimes \text{U}(1)_X$ thus implies the trace nonvanished, thereby all the families have to be taken into account which follows that the number of fermionic triplets equals to those of antitriplets. Consequently, the number of families is a integral multiple of the fundamental color number, which is three, coinciding with the observation.

There are two typical versions of the 3-3-1 models concerning respective lepton sectors. In the first version, called minimal 3-3-1 model, three $\text{SU}(3)_L$ lepton triplets take the form (ν_L, l_L, l_R^c) in which l_R are the ordinary right-handed charged-leptons [7]. In the second version, the third components of lepton triplets respectively include right-handed neutrinos, (ν_L, l_L, ν_R^c) , called 3-3-1 model with right-handed neutrinos [6]. In Refs. [10, 11], we have proposed another variant of the lepton sectors as (ν_L, l_L, N_R^c) where N_R are three new fermion singlets carrying no lepton-number in contradiction to that of the right-handed neutrinos, called 3-3-1 model with neutral fermions. Among the 3-3-1 models as mentioned, the last one can recover the tribimaximal neutrino-mixing form [12] under A_4 and S_4 flavor symmetries, respectively. For some other examples based on these flavor symmetries, see [13] and [14]. We notice also that in the minimal 3-3-1 model the charged-lepton masses can be naturally generated via the contribution of $\text{SU}(3)_L$ scalar antisextet. In the two others the neutrino masses by contrast can be arisen similarly [10, 11, 15].

The parameters of neutrino oscillations such as the squared mass differences and mixing angles are now very constrained. The data in PDG2010 [1] imply

$$\begin{aligned} s_{23}^2 &= 0.5, & s_{12}^2 &= 0.304, & s_{13}^2 &< 0.035, \\ \Delta m_{21}^2 &= 7.65 \times 10^{-5} \text{ eV}^2, & |\Delta m_{31}^2| &= 2.40 \times 10^{-3} \text{ eV}^2, \end{aligned} \quad (1)$$

where (and hereafter) the best fits are taken into accounts. Whereas, under the light of new

experiments [16], the new data [17] (see also [18] for previous fits) have been given to be slightly modified from the old fits (1):

$$\begin{aligned} s_{23}^2 &= 0.52, & s_{12}^2 &= 0.312, & s_{13}^2 &= 0.013, \\ \Delta m_{21}^2 &= 7.59 \times 10^{-5} \text{ eV}^2, & |\Delta m_{31}^2| &= 2.50 \times 10^{-3} \text{ eV}^2. \end{aligned} \quad (2)$$

If such conclusions on the mixing angles are exact, the simplest explanation is probably due to a S_3 flavor symmetry which is the smallest non-Abelian discrete group [19]. In fact, there is an approximately maximal mixing of two flavors μ and τ as given above which can be connected by a $\underline{2}$ irreducible representation of S_3 . Besides the $\underline{2}$, the group S_3 can provide two inequivalent singlet representations $\underline{1}$ and $\underline{1}'$ which play a crucial role in reproducing consistent fermion masses and mixings. The S_3 models have been studied extensively over the last decade [20].

We would like to extend the above application to the two latter 3-3-1 models with respect to the right-handed neutrinos ν_R and the neutral fermions N_R as mentioned because of the following independent issues. (i) The observed neutrino masses can be obtained by the seesaw mechanism; (ii) The anomaly cancelation as determined requires also that one family of quarks has to transform under $SU(3)_L$ differently from the two others. We should therefore search for a family symmetry with $\underline{2}$ and $\underline{3}$ representations respectively acting on such 2- and 3-family indices. Looking for a group with corresponding irreducible representations, the simplest is S_4 which has been explored in [11]. Another possibility with A_4 has been given in [10]. In this paper, it is worth to investigate a simpler group choice with S_3 in which $\underline{3}$ and $\underline{2}$ are the defining and irreducible representations of this group, respectively. The physics as we will see is different from the formers [10, 11]. It is also noted that the numbers of fermion families in the 3-3-1 model have an origin from the anomaly-free gauge symmetry naturally meet with our criteria on the dimensions of flavor group representations as such S_3 , unlike the others in the literature put by hand [13, 14, 20].

The rest of this work is as follows. In Sec. II we present the necessary elements of the 3-3-1 model with neutral fermions N_R under the S_3 symmetry as well as introducing necessary Higgs fields responsible for the quark and charged-lepton masses. Section III is devoted to the neutrino mass problem. Section IV introduces the S_3 symmetry into the 3-3-1 model with right-handed neutrinos and briefly remarks on its consequences. We summarize our results and make conclusions in Sec. V. Appendix A briefly provides the theory of S_3 group. Appendices B and C present the lepton numbers and scalar potentials of both the models, respectively.

II. THE 3-3-1 MODEL WITH NEUTRAL FERMIONS (N_R)

A. Fermion content

The gauge symmetry is given by $SU(3)_C \otimes SU(3)_L \otimes U(1)_X$ (thus named 3-3-1), where the electroweak factor $SU(3)_L \otimes U(1)_X$ is extended from those of the standard model whereas the strong interaction sector is retained. Each lepton family including a new electrically- and leptonically-neutral chiral fermion (N_R) is arranged under the $SU(3)_L$ symmetry as a triplet (ν_L, l_L, N_R^c) and a singlet l_R . The residual electric charge operator Q is related to the generators of the gauge symmetry by $Q = T_3 - \frac{1}{\sqrt{3}}T_8 + X$, where T_a ($a = 1, 2, \dots, 8$) are $SU(3)_L$ charges with $\text{Tr} T_a T_b = \frac{1}{2}\delta_{ab}$ and X is the charge of $U(1)_X$. This means that the model considered does not contain exotic electric charges in the fundamental fermion, scalar and adjoint gauge-boson representations.

The lepton number is also a residual charge and not commuting with the gauge symmetry unlike the standard model. It is better to work with a new conserved charge \mathcal{L} commuting with the gauge symmetry and related to the ordinary lepton number by diagonal matrices $L = \frac{2}{\sqrt{3}}T_8 + \mathcal{L}$ [10, 11, 21]. This is only convenient for accounting the global lepton numbers of model particles since T_8 is a gauged charge, consequently L is gauged which contrasts with outset. The T_8 can be understood as the charge of a group replication of $SU(3)_L$ but globally taken. The lepton charge arranged in this way (i.e. $L(N_R) = 0$ as assumed) will prevent unwanted interactions (due to $U(1)_{\mathcal{L}}$ symmetry) and symmetry-breakings (due to the lepton parity as shown below) providing consistent lepton and quark spectra with distinguish phenomena from those in the 3-3-1 model with right-handed neutrinos as presented in Sec. IV. By this embedding, the model also does not contain exotic leptonic charges in the fundamental fermion, scalar and adjoint gauge-boson representations, e.g. exotic quarks U, D as well as new non-Hermitian gauge bosons X^0, Y^\pm possess lepton charges as of the ordinary leptons: $L(D) = -L(U) = L(X^0) = L(Y^-) = 1$.

A brief of the theory of S_3 group is given in Appendix A. The S_3 contains one doublet irreducible representation $\underline{2}$ and two singlets $\underline{1}, \underline{1}'$. As motivated by assigning the flavor $\underline{2}$ and $\underline{3}$ contents in which the $\underline{3}$ in this case is defining representation decomposed as $\underline{3} = \underline{2} \oplus \underline{1}$, we should therefore put all the model fermions in $\underline{1}$ and $\underline{2}$. To be concrete, we put the first family fermions in $\underline{1}$, while the two other families are in $\underline{2}$. Under the $[SU(3)_L, U(1)_X, U(1)_{\mathcal{L}}, \underline{S}_3]$ symmetries as proposed the fermions correspondingly transform as follows

$$\begin{aligned}\psi_{1L} &= (\nu_{1L}, l_{1L}, N_{1R}^c)^T \sim [3, -1/3, 2/3, \underline{1}], & l_{1R} &\sim [1, -1, 1, \underline{1}], \\ \psi_{\alpha L} &= (\nu_{\alpha L}, l_{\alpha L}, N_{\alpha R}^c)^T \sim [3, -1/3, 2/3, \underline{2}], & l_{\alpha R} &\sim [1, -1, 1, \underline{2}],\end{aligned}$$

$$\begin{aligned}
Q_{1L} &= (u_{1L}, d_{1L}, U_L)^T \sim [3, 1/3, -1/3, \underline{1}], \\
u_{1R} &\sim [1, 2/3, 0, \underline{1}], \quad d_{1R} \sim [1, -1/3, 0, \underline{1}], \quad U_R \sim [1, 2/3, -1, \underline{1}], \\
Q_{\alpha L} &= (d_{\alpha L}, -u_{\alpha L}, D_{\alpha L})^T \sim [3^*, 0, 1/3, \underline{2}], \\
u_{\alpha R} &\sim [1, 2/3, 0, \underline{2}], \quad d_{\alpha R} \sim [1, -1/3, 0, \underline{2}], \quad D_{\alpha R} \sim [1, -1/3, 1, \underline{2}].
\end{aligned} \tag{3}$$

where $\alpha = 2, 3$ is a family index of the last two lepton and quark families, which are in order defined as the components of the $\underline{2}$ representations. Note that the $\underline{2}$ for quarks meets the requirement of anomaly cancelation where the last two left-quark families are in 3^* while the first one as well as the leptons are in 3 . All the \mathcal{L} charges of the model multiplets are listed in the square brackets.

In the following, we consider possibilities for generating the fermion masses. The scalar multiplets needed for this purpose would be introduced accordingly.

B. Charged-lepton mass

To generate masses for the charged leptons, we need two scalar multiplets:

$$\phi = \begin{pmatrix} \phi_1^+ \\ \phi_2^0 \\ \phi_3^+ \end{pmatrix} \sim [3, 2/3, -1/3, \underline{1}], \quad \phi' = \begin{pmatrix} \phi_1'^+ \\ \phi_2'^0 \\ \phi_3'^+ \end{pmatrix} \sim [3, 2/3, -1/3, \underline{1}'] \tag{4}$$

with VEVs $\langle \phi \rangle = (0, v, 0)^T$ and $\langle \phi' \rangle = (0, v', 0)^T$. Notice that the numbered subscripts are the indices of $SU(3)_L$. The VEV of ϕ conserves S_3 while that of ϕ' breaks this symmetry. Here the three elements of S_3 corresponding to interchanges of two within three objects are broken. Therefore the S_3 breaking in the charged lepton sector is $S_3 \rightarrow Z_3$ which consists of the identity element and the two total permutations.

The Yukawa interactions are

$$-\mathcal{L}_l = h_1 \bar{\psi}_{1L} \phi l_{1R} + h(\bar{\psi}_{2L} l_{2R} + \bar{\psi}_{3L} l_{3R}) \phi + h'(\bar{\psi}_{3L} l_{3R} - \bar{\psi}_{2L} l_{2R}) \phi' + h.c. \tag{5}$$

The mass Lagrangian reads

$$-\mathcal{L}_l^{\text{mass}} = (\bar{l}_{1L}, \bar{l}_{2L}, \bar{l}_{3L}) M_l (l_{1R}, l_{2R}, l_{3R})^T + h.c.$$

where

$$M_l = \begin{pmatrix} h_1 v & 0 & 0 \\ 0 & h v - h' v' & 0 \\ 0 & & h v + h' v' \end{pmatrix} \equiv \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & & m_\tau \end{pmatrix}, \tag{6}$$

which has the diagonal form. The diagonalization matrices are therefore $U_{lL} = U_{lR} = 1$. This means that the charged leptons $l_{1,2,3}$ by themselves are the physical mass eigenstates. The lepton mixing matrix depends on only that of the neutrinos that will be studied in the next section. The masses of muon and tau are explicitly separated by ϕ' resulting from the breaking $S_3 \rightarrow Z_3$. This is why we introduce ϕ' in accompanying with ϕ .

The experimental mass values for the charged leptons at the weak scale are given as [1]:

$$m_e = 0.511 \text{ MeV}, \quad m_\mu = 106.0 \text{ MeV}, \quad m_\tau = 1.77 \text{ GeV} \quad (7)$$

Thus, we get

$$h_1 v = 0.511 \text{ MeV}, \quad h v = 938 \text{ MeV}, \quad h' v' = 832 \text{ MeV} \quad (8)$$

It follows that $h_1 \ll h \sim h'$, provided $v' \sim v$. We particularly notice that the $\mu - \tau$ mass splitting term due to the $S_3 \rightarrow Z_3$ breaking is necessarily large like that of the S_3 conserving, which are all given in the scale of one half tau mass.

C. Quark mass

To generate the quark masses, we additionally introduce the following scalar multiplets:

$$\begin{aligned} \chi &= (\chi_1^0, \chi_2^-, \chi_3^0)^T \sim [3, -1/3, 2/3, \underline{1}], \\ \eta &= (\eta_1^0, \eta_2^-, \eta_3^0)^T \sim [3, -1/3, -1/3, \underline{1}], \\ \eta' &= (\eta_1'^0, \eta_2'^-, \eta_3'^0)^T \sim [3, -1/3, -1/3, \underline{1}']. \end{aligned} \quad (9)$$

It is noticed that these scalars do not couple to the lepton sector due to the gauge invariance. The Yukawa interactions are then

$$\begin{aligned} -\mathcal{L}_q &= f_1 \bar{Q}_{1L} \chi U_R + f \bar{Q}_L \chi^* D_R \\ &\quad + h_1^u \bar{Q}_{1L} \eta u_{1R} + h^d \bar{Q}_L \eta^* d_R + h'^d \bar{Q}_L \eta'^* d_R \\ &\quad + h_1^d \bar{Q}_{1L} \phi d_{1R} + h^u \bar{Q}_L \phi^* u_R + h'^u \bar{Q}_L \phi'^* u_R \\ &\quad + h.c. \end{aligned} \quad (10)$$

We now introduce a residual symmetry of lepton number $P_l \equiv (-1)^L$, called “lepton parity” [10, 22], in order to suppress the mixing between ordinary quarks and exotic quarks. For a summary of lepton number of the model particles, see Appendix A. The particles with even parity ($P_l = 1$)

have $L = 0, \pm 2$ such as N_R , ordinary quarks and gauge bosons, the new neutral gauge boson Z' , $\phi_{1,2}$, $\phi'_{1,2}$, $\eta_{1,2}$, $\eta'_{1,2}$, χ_3 , and so on. The odd parity particles ($P_l = -1$) possess $L = \pm 1$ such as ordinary leptons, U , D , the new non-Hermitian X and Y , ϕ_3 , ϕ'_3 , η_3 , η'_3 , $\chi_{1,2}$, and so forth. In this framework we assume that the lepton parity is an exact symmetry, not spontaneously broken. This means that η_3 , η'_3 and χ_1 cannot develop VEV, and the concerning phenomena will be skip. However, a brief discussion of broken lepton parity for the quark sector is given at the end. For the neutrino sector we always suppose, however, that the lepton parity is broken so that the neutrino mass matrix taken into account is the most general. Otherwise, the corresponding VEVs that carry odd parity will vanish. The general conclusions obtained for the lepton sector are unchanged because the lepton parity commutes with S_3 .

Suppose that the VEVs of η , η' and χ are u , u' and w , where $u = \langle \eta_1^0 \rangle$, $u' = \langle \eta_1'^0 \rangle$, $w = \langle \chi_3^0 \rangle$ (the other VEVs $\langle \eta_3^0 \rangle$, $\langle \eta_3'^0 \rangle$, and $\langle \chi_1^0 \rangle$ vanish due to the lepton parity conservation). The exotic quarks therefore get masses $m_U = f_1 w$ and $m_{D_{1,2}} = f w$. In addition, w has to be much larger than those of ϕ , ϕ' , η and η' for a consistency with the effective theory. The mass matrices for ordinary up-quarks and down-quarks are, respectively, obtained as follows:

$$\begin{aligned} M_u &= \begin{pmatrix} h_1^u u & 0 & 0 \\ 0 & h^u v + h'^u v' & 0 \\ 0 & 0 & h^u v - h'^u v' \end{pmatrix} \equiv \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_t \end{pmatrix}, \\ M_d &= \begin{pmatrix} h_1^d v & 0 & 0 \\ 0 & h^d u + h'^d u' & 0 \\ 0 & 0 & h^d u - h'^d u' \end{pmatrix} \equiv \begin{pmatrix} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{pmatrix}. \end{aligned} \quad (11)$$

In similarity to the charged lepton sector, the masses of $c-t$ and $s-b$ quarks are (in pair) separated by the scalars ϕ' and η' due to the $S_3 \rightarrow Z_3$ symmetry breaking, respectively. We see that the introduction of η' is necessary to provide the different masses of s and b quarks. The current mass values for the quarks are given by [1]

$$\begin{aligned} m_u &= (1.5 \div 3.3) \text{ MeV}, & m_d &= (3.5 \div 6.0) \text{ MeV}, & m_c &= (1.16 \div 1.34) \text{ GeV}, \\ m_s &= (70.0 \div 130.0) \text{ MeV}, & m_t &= (169.0 \div 173.3) \text{ GeV}, & m_b &= (4.13 \div 4.37) \text{ GeV}. \end{aligned} \quad (12)$$

Therefore we have

$$\begin{aligned} h_1^u u &= (1.5 \div 3.3) \text{ MeV}, & h_1^d v &= (3.5 \div 6.0) \text{ MeV}, & h^u v &= (85.08 \div 87.32) \text{ GeV}, \\ h^d u &= (2.10 \div 2.25) \text{ GeV}, & h'^u v' &= -(83.83 \div 86.07) \text{ GeV}, & h'^d u' &= -(2.00 \div 2.15) \text{ GeV}. \end{aligned} \quad (13)$$

If $u \sim v \sim u' \sim v'$, the Yukawa coupling hierarchies are $h_1^u, h_1^d \ll h^d, |h'^d| \ll h^u, |h'^u|$. It is to be noted that the S_3 breaking terms in this case are also large in comparison to the conserving ones.

The unitary matrices which couple the left-handed quarks u_L and d_L to those in the mass bases are $U_{uL} = 1$ and $U_{dL} = 1$, respectively. The CKM quark mixing matrix at the tree level is then

$$U_{\text{CKM}} = U_{dL}^\dagger U_{uL} = 1. \quad (14)$$

This is a good approximation for the realistic quark mixing matrix, which implies that the mixings among the quarks are dynamically small. The small permutations such as a breaking of the lepton parity due to the odd VEVs $\langle \eta_3^0 \rangle, \langle \eta_3'^0 \rangle, \langle \chi_1^0 \rangle$, or a violation of \mathcal{L} and/or S_3 symmetry due to unnormal Yukawa interactions, namely $\bar{Q}_{1L}\chi u_{1R}, \bar{Q}_L\chi^* d_R, \bar{Q}_{1L}\chi u_R$ and so forth, will disturb the tree level matrix resulting in mixing between ordinary and exotic quarks and possibly providing the desirable quark mixing pattern [10, 11]. This also leads to the flavor changing neutral current at the tree level but strongly suppressed [10, 11]. See also Section IV for a similar matter encountered in the 3-3-1 model with right-handed neutrinos. A detailed study on these matters are out of the scope of this work and should be skip.

III. NEUTRINO MASSES AND MIXING

The neutrino masses arise from the couplings of $\bar{\psi}_L^c \psi_L$ to scalars, where $\bar{\psi}_L^c \psi_L$ transforms as $3^* \oplus 6$ under $\text{SU}(3)_L$. Notice that in the first term of decomposition the $\psi_{1,2,3}$ are totally antisymmetric in flavor indices, while they are symmetric in the second term. For the known scalar triplets, only available interactions are $(\bar{\psi}_{2L}^c \psi_{3L} - \bar{\psi}_{3L}^c \psi_{2L})\phi'$, but explicitly suppressed because of the \mathcal{L} -symmetry. We will therefore propose a new $\text{SU}(3)_L$ antisextet instead coupling to $\bar{\psi}_L^c \psi_L$ responsible for the neutrino masses. The antisextet transforms as

$$s_i = \begin{pmatrix} s_{11}^0 & s_{12}^+ & s_{13}^0 \\ s_{12}^+ & s_{22}^{++} & s_{23}^+ \\ s_{13}^0 & s_{23}^+ & s_{33}^0 \end{pmatrix}_i \sim [6^*, 2/3, -4/3, \underline{2}], \quad (15)$$

where the numbered subscripts on the component scalars are the $\text{SU}(3)_L$ indices, whereas $i = 1, 2$ is that of S_3 . Note that i and α as mentioned belong to the same index kind. The VEV of s is set as $(\langle s_1 \rangle, \langle s_2 \rangle)$ under S_3 , in which

$$\langle s_i \rangle = \begin{pmatrix} \lambda_i & 0 & v_i \\ 0 & 0 & 0 \\ v_i & 0 & \Lambda_i \end{pmatrix}. \quad (16)$$

Following the potential minimization conditions, we have several VEV alignments. The first one is that $\langle s_1 \rangle = \langle s_2 \rangle$ then S_3 is broken into Z_2 consisting of the identity element and one interchange (within the three) of S_3 . The second one is that $\langle s_1 \rangle \neq 0 = \langle s_2 \rangle$ or $\langle s_1 \rangle = 0 \neq \langle s_2 \rangle$ then S_3 is broken into Z_3 like the case of the charged lepton sector. To obtain a realistic neutrino spectrum, in this work we argue that both the breakings $S_3 \rightarrow Z_2$ and $S_3 \rightarrow Z_3$ must be taken place. However, the VEVs of s does only one of these tasks. We therefore assume that its VEVs are aligned as the former to derive the first direction of the breakings $S_3 \rightarrow Z_2$, and this happens in any case below:

$$\lambda_1 = \lambda_2 \equiv \lambda_s, \quad v_1 = v_2 \equiv v_s, \quad \Lambda_1 = \Lambda_2 \equiv \Lambda_s. \quad (17)$$

To achieve the second direction of the breakings $S_3 \rightarrow Z_3$, we additionally introduce another scalar which lies in either $\underline{1}'$ or $\underline{2}$ (with the second alignment of VEVs as mentioned above). However, this scalar is also equivalent to break the Z_2 subgroup of the first direction. We can therefore understand the misalignment of the VEVs as follows. The S_3 is broken via two stages, the first stage is $S_3 \rightarrow Z_2$ and the second stage is $Z_2 \rightarrow \{\text{identity}\}$ (instead of $S_3 \rightarrow Z_3$). The second stage (or direction) can be achieved within each case below.

1. A new $SU(3)_L$ triplet ρ (if the \mathcal{L} -symmetry is allowed), which is impossible in $\underline{2}$ since $(\bar{\psi}_L^c \psi_L)_{\underline{2}} = (\bar{\psi}_{3L}^c \psi_{3L}, \bar{\psi}_{2L}^c \psi_{2L}) = 0$ due to antisymmetric in ψ_2 and ψ_3 , is thus put in the $\underline{1}'$:

$$\rho = \begin{pmatrix} \rho_1^+ \\ \rho_2^0 \\ \rho_3^+ \end{pmatrix} \sim [3, 2/3, -4/3, \underline{1}'], \quad (18)$$

with the VEVs given by $\langle \rho \rangle = (0, v_\rho, 0)^T$.

2. Another antisextet s' , which is impossible in $\underline{1}'$ since $(\bar{\psi}_L^c \psi_L)_{\underline{1}'} = \bar{\psi}_{2L}^c \psi_{3L} - \bar{\psi}_{3L}^c \psi_{2L} = 0$ due to symmetric in ψ_2 and ψ_3 , is thus left with the $\underline{2}$ with VEVs chosen by

$$s'_i = \begin{pmatrix} s'_{11} & s'_{12} & s'_{13} \\ s'_{12} & s'_{22} & s'_{23} \\ s'_{13} & s'_{23} & s'_{33} \end{pmatrix}_i \sim [6^*, 2/3, -4/3, \underline{2}], \quad \langle s'_1 \rangle = \begin{pmatrix} \lambda'_s & 0 & v'_s \\ 0 & 0 & 0 \\ v'_s & 0 & \Lambda'_s \end{pmatrix}, \quad \langle s'_2 \rangle = 0. \quad (19)$$

In calculation, combining both cases we have the Yukawa interactions:

$$\begin{aligned} -\mathcal{L}_\nu &= \frac{1}{2}x(\bar{\psi}_{2L}^c \psi_{2L} s_1 + \bar{\psi}_{3L}^c \psi_{3L} s_2) + \frac{1}{2}y\bar{\psi}_{1L}^c(\psi_{2L} s_2 + \psi_{3L} s_1) \\ &\quad + \frac{1}{2}x'(\bar{\psi}_{2L}^c \psi_{2L} s'_1 + \bar{\psi}_{3L}^c \psi_{3L} s'_2) + \frac{1}{2}y'\bar{\psi}_{1L}^c(\psi_{2L} s'_2 + \psi_{3L} s'_1) \\ &\quad + \frac{1}{2}z(\bar{\psi}_{2L}^c \psi_{3L} - \bar{\psi}_{3L}^c \psi_{2L})\rho \\ &\quad + h.c., \end{aligned} \quad (20)$$

where the couplings y , y' and z are of lepton flavor changing interactions. The mass Lagrangian for the neutrinos is given by

$$\begin{aligned}
-\mathcal{L}_\nu^{\text{mass}} = & \frac{1}{2}x(\lambda_s\bar{\nu}_{2L}^c\nu_{2L} + v_s\bar{\nu}_{2L}^cN_{2R}^c + v_s\bar{N}_{2R}\nu_{2L} + \Lambda_s\bar{N}_{2R}N_{2R}^c) \\
& + \frac{1}{2}x(\lambda_s\bar{\nu}_{3L}^c\nu_{3L} + v_s\bar{\nu}_{3L}^cN_{3R}^c + v_s\bar{N}_{3R}\nu_{3L} + \Lambda_s\bar{N}_{3R}N_{3R}^c) \\
& + \frac{1}{2}y(\lambda_s\bar{\nu}_{1L}^c\nu_{2L} + v_s\bar{\nu}_{1L}^cN_{2R}^c + v_s\bar{N}_{1R}\nu_{2L} + \Lambda_s\bar{N}_{1R}N_{2R}^c) \\
& + \frac{1}{2}y(\lambda_s\bar{\nu}_{1L}^c\nu_{3L} + v_s\bar{\nu}_{1L}^cN_{3R}^c + v_s\bar{N}_{1R}\nu_{3L} + \Lambda_s\bar{N}_{1R}N_{3R}^c) \\
& + \frac{1}{2}x'(\lambda'_s\bar{\nu}_{2L}^c\nu_{2L} + v'_s\bar{\nu}_{2L}^cN_{2R}^c + v'_s\bar{N}_{2R}\nu_{2L} + \Lambda'_s\bar{N}_{2R}N_{2R}^c) \\
& + \frac{1}{2}y'(\lambda'_s\bar{\nu}_{1L}^c\nu_{3L} + v'_s\bar{\nu}_{1L}^cN_{3R}^c + v'_s\bar{N}_{1R}\nu_{3L} + \Lambda'_s\bar{N}_{1R}N_{3R}^c) \\
& + \frac{1}{2}zv_\rho(-\bar{\nu}_{2L}^cN_{3R}^c + \bar{N}_{2R}\nu_{3L} + \bar{\nu}_{3L}^cN_{2R}^c - \bar{N}_{3R}\nu_{2L}) \\
& + h.c.
\end{aligned} \tag{21}$$

We can rewrite

$$-\mathcal{L}_\nu^{\text{mass}} = \frac{1}{2}\bar{\chi}_L^c M_\nu \chi_L + h.c., \quad \chi_L \equiv \begin{pmatrix} \nu_L \\ N_R^c \end{pmatrix}, \quad M_\nu \equiv \begin{pmatrix} M_L & M_D^T \\ M_D & M_R \end{pmatrix}, \tag{22}$$

where $\nu_L = (\nu_{1L}, \nu_{2L}, \nu_{3L})^T$, $N_R = (N_{1R}, N_{2R}, N_{3R})^T$ and

$$M_{L,R,D} = \begin{pmatrix} 0 & b_{1L,R,D} & b_{2L,R,D} \\ b_{1L,R,D} & c_{1L,R,D} & d_{L,R,D} \\ b_{2L,R,D} & -d_{L,R,D} & c_{2L,R,D} \end{pmatrix}, \tag{23}$$

where

$$\begin{aligned}
b_{1L} &= \frac{\lambda_s y}{2}, & b_{1D} &= \frac{v_s y}{2}, & b_{1R} &= \frac{\Lambda_s y}{2}, \\
b_{2L} &= \frac{(\lambda_s y + \lambda'_s y')}{2}, & b_{2D} &= \frac{(v_s y + v'_s y')}{2}, & b_{2R} &= \frac{(\Lambda_s y + \Lambda'_s y')}{2}, \\
c_{1L} &= (\lambda_s x + \lambda'_s x'), & c_{1D} &= (v_s x + v'_s x'), & c_{1R} &= (\Lambda_s x + \Lambda'_s x'), \\
c_{2L} &= \lambda_s x, & c_{2D} &= v_s x, & c_{2R} &= \Lambda_s x, \\
d_L &= d_R = 0, & d_D &= zv_\rho \equiv d.
\end{aligned}$$

The remarks are in order

- The VEVs with even lepton-parity are: λ_s , λ'_s , Λ_s and Λ'_s ;
- The VEVs with odd lepton-parity are: v_s , v'_s and v_ρ .

If the lepton parity is conserved we have $M_D = 0$ since $v_s = v'_s = v_\rho = 0$. There is no mixing between the left-handed neutrinos and the neutral fermions. The observed neutrinos are just ν_L with masses given by M_L consisting of λ_s and λ'_s VEVs which are naturally small as given in eV order in similarity to the case of the standard model with scalar triplets [23] (called type II seesaw mechanism [24]). However, this situation as we see below cannot fit the data provided that the contribution of s' is as a small perturbation.

In general with the combined cases and lepton parity breaking, three observed neutrinos gain masses via a cooperation of type I and type II seesaw mechanisms derived from (22) as

$$M_{\text{eff}} = M_L - M_D^T M_R^{-1} M_D = \begin{pmatrix} A & B_1 & B_2 \\ B_1 & C_1 & \mathcal{D} \\ B_2 & \mathcal{D} & C_2 \end{pmatrix}, \quad (24)$$

where

$$\begin{aligned} A &= -(b_{1R}b_{2D} - b_{1D}b_{2R})^2 / (b_{2R}^2c_{1R} + b_{1R}^2c_{2R}), \\ B_1 &= [b_{1L}c_{1R}b_{2R}^2 + b_{1L}c_{2R}b_{1R}^2 + b_{1R}b_{2R}b_{2D}c_{1D} - b_{1D}c_{1D}b_{2R}^2 - b_{1D}b_{2D}c_{1R}b_{2R} \\ &\quad - b_{1R}c_{2R}b_{1D}^2 + db_{1R}(b_{1R}b_{2D} - b_{2R}b_{1D})] / (b_{2R}^2c_{1R} + b_{1R}^2c_{2R}), \\ B_2 &= [b_{2L}c_{1R}b_{2R}^2 + b_{2L}c_{2R}b_{1R}^2 + b_{1R}b_{2R}b_{1D}c_{2D} - b_{2D}c_{2D}b_{1R}^2 - b_{1D}b_{2D}c_{2R}b_{1R} \\ &\quad - b_{2R}c_{1R}b_{2D}^2 + db_{2R}(b_{1R}b_{2D} - b_{2R}b_{1D})] / (b_{2R}^2c_{1R} + b_{1R}^2c_{2R}), \\ C_1 &= [b_{1D}^2c_{1R}c_{2R} + b_{2R}^2c_{1L}c_{1R} + b_{1R}^2c_{1L}c_{2R} - 2b_{1D}c_{1D}b_{1R}c_{2R} - c_{1D}^2b_{2R}^2 \\ &\quad - 2db_{2R}(b_{1R}c_{1D} - b_{1D}c_{1R}) - d^2b_{1R}^2] / (b_{2R}^2c_{1R} + b_{1R}^2c_{2R}), \\ C_2 &= [b_{2D}^2c_{1R}c_{2R} + b_{1R}^2c_{2L}c_{2R} + b_{2R}^2c_{1R}c_{2L} - 2b_{2D}c_{2D}b_{2R}c_{1R} - c_{2D}^2b_{1R}^2 \\ &\quad - 2db_{1R}(b_{2D}c_{2R} - b_{2R}c_{2D}) - d^2b_{2R}^2] / (b_{2R}^2c_{1R} + b_{1R}^2c_{2R}), \\ \mathcal{D} &= [(b_{1D}c_{1R} - b_{1R}c_{1D})(b_{2D}c_{2R} - b_{2R}c_{2D}) + d(b_{2D}b_{2R}c_{1R} - b_{1D}b_{1R}c_{2R} \\ &\quad + b_{1R}^2c_{2D} - b_{2R}^2c_{1D}) - d^2b_{1R}b_{2R}] / (b_{2R}^2c_{1R} + b_{1R}^2c_{2R}). \end{aligned} \quad (25)$$

The comments are in order:

- If the subgroup Z_2 is unbroken, we have $A = \mathcal{D} = 0$, $B_1 = B_2$ and $C_1 = C_2$;
- If the Z_2 is broken by only the case 1, we have $A = 0$, $B_1 = B_2$, $C_1 = C_2$, but $\mathcal{D} \neq 0$;
- If the Z_2 is broken by both the cases, but the case 2 is regarded as a small perturbation, we have $A \approx 0$, $B_1 \approx B_2$, $C_1 \approx C_2$, and $\mathcal{D} \neq 0$.

In addition, it is able to introduce one more antisextet in $\underline{1}$ which does not break the subgroup Z_2 and implies $A \neq 0$ for contribution to our results above, but this does not change our conclusions below and should be skip without loss of generality.

We next divide our considerations into two cases to fit the data: the first case is only case 1 above, and the second case is a combination of the both.

A. Experimental constraints under case 1 only

In the case 1, $\lambda'_s = v'_s = \Lambda'_s = 0$, we have $A = 0$, $B_1 = B_2 \equiv B$, $C_1 = C_2 \equiv C$, $D \equiv D \neq 0$, and

$$M_{\text{eff}} = \begin{pmatrix} 0 & B & B \\ B & C & D \\ B & D & C \end{pmatrix}, \quad (26)$$

where

$$B = \left(\lambda_s - \frac{v_s^2}{\Lambda_s} \right) \frac{y}{2}, \quad C = \left(\lambda_s - \frac{v_s^2}{\Lambda_s} \right) x - \frac{v_\rho^2}{\Lambda_s} \frac{z^2}{2x}, \quad D = -\frac{v_\rho^2}{\Lambda_s} \frac{z^2}{2x}. \quad (27)$$

This mass matrix takes the form similar to that of the unbroken Z_2 (with $v_\rho = 0$). However, the breaking of Z_2 ($v_\rho \neq 0$, thus $D \neq 0$) in this case is necessary to fit the data (see below). Indeed, we can diagonalize $U^T M_{\text{eff}} U = \text{diag}(m_1, m_2, m_3)$ where

$$\begin{aligned} m_1 &= \frac{1}{2} \left(C + D - \sqrt{8B^2 + (C + D)^2} \right), \\ m_2 &= \frac{1}{2} \left(C + D + \sqrt{8B^2 + (C + D)^2} \right), \\ m_3 &= C - D, \end{aligned} \quad (28)$$

and the corresponding eigenstates put in the lepton mixing matrix:

$$U = \begin{pmatrix} -m_2/\sqrt{m_2^2 + 2B^2} & -m_1/\sqrt{m_1^2 + 2B^2} & 0 \\ B/\sqrt{m_2^2 + 2B^2} & B/\sqrt{m_1^2 + 2B^2} & -\frac{1}{\sqrt{2}} \\ B/\sqrt{m_2^2 + 2B^2} & B/\sqrt{m_1^2 + 2B^2} & \frac{1}{\sqrt{2}} \end{pmatrix}. \quad (29)$$

Note that $m_1 m_2 = -2B^2$. This matrix can be parameterized in three Euler's angles, which implies:

$$\theta_{13} = 0, \quad \theta_{23} = \pi/4, \quad \tan \theta_{12} = \sqrt{-\frac{m_1}{m_2}}. \quad (30)$$

This case coincides with the data since $\theta_{13} < \pi/13$ and $\theta_{23} \simeq \pi/4$ [1] and close to the proposal of the tribimaximal neutrino-mixing form [12]. For the remaining constraints, taking the central

values from the data [1] as displayed in (1), $t_{12}^2 \simeq 0.435$ (i.e. $s_{12}^2 = 0.304$), $\Delta m_{21}^2 = 7.65 \times 10^{-5} \text{ eV}^2$ and $|\Delta m_{31}^2| = 2.4 \times 10^{-3} \text{ eV}^2$, we have a solution $m_1 = -0.42 \times 10^{-2} \text{ eV}$, $m_2 = 0.97 \times 10^{-2} \text{ eV}$ and $m_3 = 4.9 \times 10^{-2} \text{ eV}$ (normal ordering). It follows $B \simeq 0.451 \times 10^{-2} \text{ eV}$, $C \simeq 2.725 \times 10^{-2} \text{ eV}$ and $D \simeq -2.175 \times 10^{-2} \text{ eV}$ (as expected). Now, it is natural to choose λ_s , v_s^2/Λ_s and v_ρ^2/Λ_s in eV order. From (27), we can find the three parameters x , y and z , with the provided B , C and D , respectively. It is noteworthy that in this case the contribution of the Z_2 (or $S_3 \rightarrow Z_3$) breaking parameter ($\sim v_\rho^2/\Lambda_s$) transforming under $\underline{1}'$ to the neutrino masses is comparable to that of the $S_3 \rightarrow Z_2$ one ($\sim \lambda_s$, v_s^2/Λ_s). This makes the model viable and in some sense quite similar to those of the quark and charged-lepton sectors.

The recent considerations have implied $\theta_{13} \neq 0$ [16, 18], but small as given in (2). If it is correct, this case will fail. But, a combination of the case 1 with the case 2 above improves this.

B. Experimental constraints under combination of case 1 and case 2

In a scenario where both the case 1 and case 2 are taken place, the neutrino mass matrix (24) can be rewritten in the form:

$$M_{\text{eff}} = \begin{pmatrix} 0 & B & B \\ B & C & D \\ B & D & C \end{pmatrix} + \begin{pmatrix} \epsilon & p_1 & p_2 \\ p_1 & q_1 & r \\ p_2 & r & q_2 \end{pmatrix}, \quad (31)$$

where B , C and D are given by (27) accommodated in the first term or matrix due to the contribution of the scalar antisextet s and triplet ρ only as in the case 1 (26). The last matrix is a deviation from the contribution of the case 1 due to the scalar antisextet s' , namely $\epsilon \equiv A$, $p_{1,2} = B_{1,2} - B$, $q_{1,2} = C_{1,2} - C$ and $r = \mathcal{D} - D$, with the A , $B_{1,2}$, $C_{1,2}$ and \mathcal{D} being defined in (25). Indeed, if the case 2 or the s' contribution is forbidden, the deviations ϵ , p , q , r will vanish (to be concrete, see below), therefore the mass matrix M_{eff} (31) reduces to its first term coinciding with the case 1. The first term as shown can approximately fit the new data [18] with a “small” deviation for θ_{13} . The second term proportional to ϵ , p , q , r due to contribution of the antisextet s' will take the role for such a deviation of θ_{13} . So, in this case we consider the s' contribution as a small perturbation and terminating the theory at the first order.

Provided that $\langle s' \rangle \ll \langle s \rangle$ or $\lambda'_s/\lambda_s \sim v'_s/v_s \sim \Lambda'_s/\Lambda_s \ll 1$, one can evaluate

$$\epsilon \simeq -\frac{y'^2 v_s^2}{8x \Lambda_s} \left(\frac{v'_s}{v_s} - \frac{\Lambda'_s}{\Lambda_s} \right)^2 \ll 1, \quad (32)$$

which lies at the second order of the perturbation, thus ignored. The remaining parameters $p_{1,2}$, $q_{1,2}$, r are easily obtained as follows

$$\begin{aligned}
p_1 &\simeq \frac{1}{16x\Lambda_s} \left\{ \left[\Lambda_s \lambda_s (x'y + 2xy') - \frac{v_s^2 x' (3x + 2y)}{3} + v_s v_\rho y' z \right] \frac{\Lambda'_s}{\Lambda_s} + \frac{v_s}{3} (v_s x'y + 3v_\rho y' z) \frac{v'_s}{v_s} \right\}, \\
p_2 &\simeq \frac{1}{16x\Lambda_s} \left\{ \left[\Lambda_s \lambda_s (x'y + 2xy') - \frac{v_s^2 x' (x - 3y)}{3} + v_s v_\rho y' z \right] \frac{\Lambda'_s}{\Lambda_s} + \Lambda_s \lambda_s x (x' + y') \frac{\lambda'_s}{\lambda_s} \right. \\
&\quad \left. + \frac{v_s}{3} y' (3v_\rho z - v_s x) \frac{v'_s}{v_s} \right\}, \\
q_1 &\simeq \frac{1}{8x\Lambda_s} \left\{ \left[\frac{v_s^2 x}{y} (x'y - 2xy') + \frac{\Lambda_s \lambda_s x}{y} (x'y + 2xy') + 2v_s v_\rho x' z \right] \frac{\Lambda'_s}{\Lambda_s} + \Lambda_s \lambda_s x' (x + x') \frac{\lambda'_s}{\lambda_s} \right. \\
&\quad \left. - 2v_s x' [v_s (x + z) + v_\rho z] \frac{v'_s}{v_s} \right\}, \\
q_2 &\simeq \frac{1}{8\Lambda_s} \left\{ \left[\Lambda_s \lambda_s \frac{(x'y + 2xy')}{y} + \frac{v_s y'}{y} (4v_\rho z - v_s x) - \frac{2v_\rho^2 y' z^2}{xy} \right] \frac{\Lambda'_s}{\Lambda_s} + \frac{v_s y'}{y} (v_s x - 4zv_\rho) \frac{v'_s}{v_s} \right\}, \\
r &\simeq \frac{1}{8x\Lambda_s} \frac{v_\rho z}{y} \left\{ [\lambda_s (x'y - xy') - v_\rho y' z] \frac{\Lambda'_s}{\Lambda_s} + v_s (xy' - x'y) \frac{v'_s}{v_s} \right\}, \tag{33}
\end{aligned}$$

which all start from the first order of the perturbation. *It is noteworthy that the presence of ρ is important since by contrast the r will start from the second order and ignored as ϵ . The resulting mass matrix cannot fit the new data.*

The explicit form of the mass matrix (31) is thus given by

$$M_{\text{eff}} = \begin{pmatrix} 0 & B & B \\ B & C & D \\ B & D & C \end{pmatrix} + \frac{\lambda'_s}{\lambda_s} M_\lambda + \frac{v'_s}{v_s} M_v + \frac{\Lambda'_s}{\Lambda_s} M_\Lambda, \tag{34}$$

where the last three terms are the perturbative contributions at the first order with

$$\begin{aligned}
M_\lambda &\equiv \frac{1}{8x} \begin{pmatrix} 0 & 0 & \frac{\lambda_s x (x' + y')}{2} \\ 0 & \lambda_s x' (x + x') & 0 \\ \frac{\lambda_s x (x' + y')}{2} & 0 & \lambda_s x' (x + x') \end{pmatrix}, \\
M_v &\equiv \frac{v_s}{8x\Lambda_s} \begin{pmatrix} 0 & \frac{1}{6} (v_s x'y + 3v_\rho y' z) & \frac{1}{6} y' (3v_\rho z - v_s x) \\ \frac{1}{6} (v_s x'y + 3v_\rho y' z) & 2x' [v_s (x + z) + v_\rho z] & \frac{v_\rho z}{y} (xy' - x'y) \\ \frac{1}{6} y' (3v_\rho z - v_s x) & \frac{v_\rho z}{y} (xy' - x'y) & \frac{y'}{y} (v_s x - 4zv_\rho) \end{pmatrix}, \\
M_\Lambda &\equiv \frac{1}{8x\Lambda_s} \begin{pmatrix} 0 \\ \frac{1}{2} \left[\Lambda_s \lambda_s (x'y + 2xy') - \frac{v_s^2 x' (3x + 2y)}{3} + v_s v_\rho y' z \right] \\ \frac{1}{2} \left[\Lambda_s \lambda_s (x'y + 2xy') - \frac{v_s^2 x' (x - 3y)}{3} + v_s v_\rho y' z \right] \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
& \frac{1}{2} \left[\Lambda_s \lambda_s (x'y + 2xy') - \frac{v_s^2 x' (3x+2y)}{3} + v_s v_\rho y' z \right] \\
& \frac{v_s^2 x}{y} (x'y - 2xy') + \frac{\Lambda_s \lambda_s}{y} (x'y + 2xy') + 2v_s v_\rho x' z \\
& \frac{v_\rho z}{y} [\lambda_s (x'y - xy') - v_\rho y' z] \\
& \frac{1}{2} \left[\Lambda_s \lambda_s (x'y + 2xy') - \frac{v_s^2 x' (x-3y)}{3} + v_s v_\rho y' z \right] \\
& \frac{v_\rho z}{y} [\lambda_s (x'y - xy') - v_\rho y' z] \\
& \Lambda_s \lambda_s \frac{(x'y+2xy')}{y} + \frac{v_s y'}{y} (4v_\rho z - v_s x) - \frac{2v_\rho^2 y' z^2}{xy} \Bigg), \tag{35}
\end{aligned}$$

which all are in the same order with the dominant contributions B, C, D proportional to the mass scale of observed neutrinos $\lambda_s, v_s^2/\Lambda_s, v_\rho^2/\Lambda_s, v_s v_\rho/\Lambda_s$.

The physical neutrino masses at the first order are obtained as

$$\begin{aligned}
m'_{1,2} &= m_{1,2} + \frac{B^2}{(m_{2,1})^2 + 2B^2} \frac{1}{8xy\Lambda_s} \left(K_{1,2}^1 \frac{\Lambda'_s}{\Lambda_s} + K_{1,2}^2 \frac{\lambda'_s}{\lambda_s} + K_{1,2}^3 \frac{v'_s}{v_s} \right), \\
m'_3 &= m_3 + \frac{1}{8xy\Lambda_s} \left(K_3^1 \frac{\Lambda'_s}{\Lambda_s} + K_3^2 \frac{\lambda'_s}{\lambda_s} + K_3^3 \frac{v'_s}{v_s} \right), \tag{36}
\end{aligned}$$

where

$$\begin{aligned}
K_1^1 &= -3v_\rho^2 y' z^2 + v_s^2 x (x'y - 3xy') + 2v_\rho \lambda_s z (x'y - xy') + 2v_s v_\rho z (x'y + 2xy') \\
&+ 2\lambda_s \Lambda_s x (x'y + 2xy') - \frac{2m_2}{B} [\Lambda_s \lambda_s y (x'y + 2xy') - \frac{v_s^2 x' y (4x - y)}{6} + v_s v_\rho y y' z], \\
K_1^2 &= \Lambda_s \lambda_s y \left[x' (x + x') - \frac{m_2}{B} x (x + x') \right], \\
K_1^3 &= v_s [-2v_s x' y (x + z) + v_\rho z (-3yx' + xy')] - \frac{m_2}{B} \frac{v_s y}{3} [v_s (x'y - xy') + 6v_\rho y' z], \\
K_3^1 &= -\frac{v_\rho^2 y' z^2}{2} + \frac{v_s^2 x (x'y - 3xy')}{2} + v_\rho \lambda_s z (-x'y + xy') + v_s v_\rho z (x'y + 2xy') \\
&+ \lambda_s \Lambda_s x (x'y + 2xy'), \\
K_3^2 &= \Lambda_s \lambda_s x' y (x + x'), \quad K_3^3 = -v_s [2v_s x' y (x + z) + v_\rho z (x'y + xy')], \tag{37}
\end{aligned}$$

with K_2^a ($a = 1, 2, 3$) similarly given as K_1^a but m_2 is replaced by m_1 . The $m_{1,2,3}$ are the mass values as of the case 1 given by (28). For the corresponding perturbed eigenstates, we put

$$U \longrightarrow U' = U + \Delta U, \tag{38}$$

where U is defined by (29) as of the case 1 and

$$\Delta U = \begin{pmatrix} -\frac{m_1}{\sqrt{m_1^2+2B^2}} F_1 & \frac{m_2}{\sqrt{m_2^2+2B^2}} F_1 & \frac{m_1}{\sqrt{m_1^2+2B^2}} F_3 - \frac{m_2}{m_1^2+2B^2} F_2 \\ \frac{B}{\sqrt{m_1^2+2B^2}} F_1 + \frac{F_2}{2} & -\frac{B}{\sqrt{m_2^2+2B^2}} F_1 - \frac{F_3}{2} & \frac{B}{\sqrt{m_2^2+2B^2}} F_2 - \frac{B}{\sqrt{m_1^2+2B^2}} F_3 \\ \frac{B}{\sqrt{m_1^2+2B^2}} F_1 - \frac{F_2}{2} & -\frac{B}{\sqrt{m_2^2+2B^2}} F_1 + \frac{F_3}{2} & \frac{B}{\sqrt{m_2^2+2B^2}} F_2 - \frac{B}{\sqrt{m_1^2+2B^2}} F_3 \end{pmatrix}, \tag{39}$$

with

$$\begin{aligned}
F1 &= \frac{(m_1 + m_2)(p_1 + p_2) - B(q_1 + q_2 + 2r)}{\sqrt{2}(m_1 - m_2)^2}, \\
F2 &= \frac{m_2(p_2 - p_1) + B(q_1 - q_2)}{(m_1 - m_3)\sqrt{m_2^2 + 2B^2}}, \\
F3 &= \frac{m_1(p_2 - p_1) + B(q_1 - q_2)}{(m_2 - m_3)\sqrt{m_1^2 + 2B^2}}.
\end{aligned} \tag{40}$$

The lepton mixing matrix in this case U' can still be parameterized in three new Euler's angles θ'_{ij} , which are also a perturbation from the θ_{ij} in the case 1, defined by

$$\begin{aligned}
s_{13}'^2 &= \left| \sqrt{\frac{1}{1 + \alpha^2}}(\alpha F_3 - F_2) \right|^2, \\
t_{12}'^2 &= \left| \alpha \left(1 - \frac{1 + \alpha^2}{\alpha} F_1 \right) \right|^2, \\
t_{23}'^2 &= \left| \frac{1}{\sqrt{2}} - 2\sqrt{\frac{1}{1 + \alpha^2}}(\alpha F_2 - F_3) \right|^2,
\end{aligned}$$

where $\alpha = \sqrt{-\frac{m_1}{m_2}}$ is just the t_{12} of the case 1.

It is easily shown that our model is consistent since the five experimental constraints on the mixing angles and squared mass differences of neutrinos can be respectively fitted with the five Yukawa coupling parameters x, y, x', y', z of the s, s' antisextets and ρ triplet scalars, provided that the VEVs are previously given. Indeed, let us first assume $\lambda_s = v_s^2/\Lambda_s = v_s v_\rho/\Lambda_s = v_\rho^2/\Lambda_s = 1$ eV and $\lambda'_s/\lambda_s = v'_s/v_s = \Lambda'_s/\Lambda_s = 0.1$ which are safely small. Taking the new data (2): $s_{13}'^2 = 0.013$, $s_{12}'^2 = 0.312$, $s_{23}'^2 = 0.52$ as well as $\Delta m_{21}'^2 = 7.59 \times 10^{-5}$ eV² and $|\Delta m_{31}'^2| = 2.5 \times 10^{-3}$ eV², we obtain $x \simeq 0.049$, $y \simeq 0.00895$, $x' \simeq 0.00149$, $y' \simeq 9.02 \times 10^{-4}$ and $z \simeq 0.0461$. The neutrino masses are explicitly given as $m'_1 = -0.41 \times 10^{-2}$ eV, $m'_2 = 0.97 \times 10^{-2}$ eV and $m'_3 = 4.9 \times 10^{-2}$ eV which are in a normal ordering.

C. Remark on breaking, VEVs and rho parameter

Both the fitting cases mentioned above require $\mathcal{D} \neq 0$. Hence, to have a consistent neutrino spectrum we conclude that the lepton parity must be broken because by contrast \mathcal{D} vanishes. Also, both the directions $S_3 \rightarrow Z_2$ and $S_3 \rightarrow Z_3$ must be taken place.

We remark that for both the fitting cases the seesaw scale Λ_s is not needed to be so large that can naturally be taken at TeV scale as the VEV ω of χ . This is because v_s and v_ρ carry lepton number,

simultaneously breaking the lepton parity which are naturally constrained to be much smaller than the electroweak scale [10, 11, 25]. This is also behind a theoretical fact that ω , Λ_s are scales for the gauge symmetry breaking in a stage from $SU(3)_L \otimes U(1)_X \rightarrow SU(2)_L \otimes U(1)_Y$. They will provide masses for the new gauge bosons such as Z' , X and Y . Also, the exotic quarks gain masses from ω while the neutral fermions are from Λ_s . The second stage of the gauge symmetry breaking is from $SU(2)_L \otimes U(1)_Y \rightarrow U(1)_Q$ achieved by the electroweak scale VEVs such as u, u', v, v' responsible for ordinary particle masses. In combination with those of type II seesaw as determined, the hierarchies in VEVs are summarized as

$$eV \sim \lambda_s < v_s, \quad v_\rho < v, \quad v', \quad u, \quad u' < \Lambda_s, \quad \omega \sim \text{TeV}. \quad (41)$$

Here the VEVs of s' as the role of perturbation, $\lambda'_s/\lambda_s \sim v'_s/v_s \sim \Lambda'_s/\Lambda_s \ll 1$, are not mentioned.

Our model contains a lot of $SU(2)_L$ scalar doublets and triplets that may modify the precision electroweak data. The most serious one can result from the tree-level contributions to the ρ parameter. To see this let us approximate W mass and ρ :

$$m_W^2 = \frac{g^2}{2} v_w^2, \quad \rho = \frac{m_W^2}{c_W^2 m_Z^2} \simeq 1 - \frac{2\lambda_s^2}{v_w^2}, \quad (42)$$

where $v_w^2 \simeq (u^2 + u'^2 + v^2 + v'^2) = (174 \text{ GeV})^2$ is naturally given due to (41) and $\langle \chi_1^0 \rangle \ll u, u', v, v'$ by the same reason as v_ρ, v_s . Because λ_s is in eV scale responsible for the observed neutrino masses, the ρ is absolutely close to one and in agreement with the data [1].

IV. S_3 SYMMETRY IN THE 3-3-1 MODEL WITH RIGHT-HANDED NEUTRINOS (ν_R)

The fermion content of this model can be given as

$$\begin{aligned} \psi_{1L} &= (\nu_{1L}, l_{1L}, \nu_{1R}^c)^T \sim [3, -1/3, 1/3, \underline{1}], & l_{1R} &\sim [1, -1, 1, \underline{1}], \\ \psi_{\alpha L} &= (\nu_{\alpha L}, l_{\alpha L}, \nu_{\alpha R}^c)^T \sim [3, -1/3, 1/3, \underline{2}], & l_{\alpha R} &\sim [1, -1, 1, \underline{2}], \\ Q_{1L} &= (u_{1L}, d_{1L}, U_L)^T \sim [3, 1/3, -2/3, \underline{1}], \\ u_{1R} &\sim [1, 2/3, 0, \underline{1}], & d_{1R} &\sim [1, -1/3, 0, \underline{1}], & U_R &\sim [1, 2/3, -2, \underline{1}], \\ Q_{\alpha L} &= (d_{\alpha L}, -u_{\alpha L}, D_{\alpha L})^T \sim [3^*, 0, 2/3, \underline{2}], \\ u_{\alpha R} &\sim [1, 2/3, 0, \underline{2}], & d_{\alpha R} &\sim [1, -1/3, 0, \underline{2}], & D_{\alpha R} &\sim [1, -1/3, 2, \underline{2}]. \end{aligned} \quad (43)$$

Here the difference from the previous model is that $L(\nu_R) = 1$. Hence the exotic quarks and new non-Hermitian gauge bosons are bilepton: $L(D) = -L(U) = L(X^0) = L(Y^-) = 2$. And, the leptonic operator is given by $L = \frac{4}{\sqrt{3}} T_8 + \mathcal{L}$ with \mathcal{L} listed in the square brackets.

The scalar sector in this model is similar to, but simpler than the previous model since the ρ triplet is not necessary to be introduced. This is due to the fact that the interaction $(\bar{\psi}_{2L}^c \psi_{3L} - \bar{\psi}_{3L}^c \psi_{2L})\phi'$ is allowed (since \mathcal{L} is conserved), thus the ϕ' triplet can play the role instead of ρ as in the case 1 above. Notice that the \mathcal{L} charges of all the scalars in this model are changed while the other charges are remained:

$$\begin{aligned}\phi &\sim [3, 2/3, -2/3, \underline{1}], & \phi' &\sim [3, 2/3, -2/3, \underline{1}'] \\ \eta &\sim [3, -1/3, -2/3, \underline{1}], & \eta' &\sim [3, -1/3, -2/3, \underline{1}'], & \chi &\sim [3, -1/3, 4/3, \underline{1}], \\ s &\sim [6^*, 2/3, -2/3, \underline{2}], & s' &\sim [6^*, 2/3, -2/3, \underline{2}].\end{aligned}\tag{44}$$

The charged-lepton and quark masses are similar to the pervious model. However, the lepton parity in this case does not work. Exactly, it cannot suppress the mixing between ordinary quarks and exotic quarks since the odd fields as mentioned in the previous model are now even $-L(\eta_3^0) = -L(\eta_3'^0) = L(\chi_1^0) = 2$ which can develop VEVs. The lepton numbers of particles in this model are listed in App. B. This mixing can only be prevented if we suppose that the L charge is not spontaneously broken. However, this discards the seesaw mechanism since $\lambda_s = \lambda'_s = \Lambda_s = \Lambda'_s = 0$. The neutrinos have only Dirac masses via the VEVs v' of ϕ' , v_s and v'_s which are not natural in the same simple extension of the standard model with Dirac neutrino masses [26].

Let us recall that it is different from the previous model since in that case we have no mixing between the two kinds of quarks due to the lepton parity while L is still broken responsible for the neutrino masses via the type I seesaw mechanism where $v_\rho = v_s = v'_s = 0$. However, as mentioned it is not realistic under the data when the s' contribution is regarded as a small perturbation. In contradiction, if the s' contribution becomes comparable, the situation will change (which has not been considered in the present work).

All the issues above can be resolved by imposing a spontaneous symmetry breaking of L . This breaking can be explicitly derived via a L -violating scalar potential. It also proves that the VEVs λ_s , v_s^2/Λ_s , $v_s v'/\Lambda_s$, v'^2/Λ_s responsible for the observed neutrino masses are naturally small [10, 11]. The results on the neutrino masses and mixings are given in similarity to the previous model with the replacement of v_ρ by v' . The mixing between the exotic quarks and ordinary quarks at the tree level happens but small for the same reason. The flavor changing neutral current starts from the tree level but strongly suppressed [10, 11]. However, the difference from the previous model is that since the v' and v_s belong to the electroweak scale the seesaw scale Λ_s is not needed to be in TeV order. In principle it can flip up to a very high scale such as the GUT one.

V. CONCLUSIONS

We have argued that the 3-3-1 models may accommodate the seesaw mechanisms naturally. In fact, the right-handed neutrinos or neutral fermions can exist as basic objects needed to complete multiplets extended from those of the standard model. We have shown that the TeV seesaw mechanism can be naturally obtained in the 3-3-1 model with neutral fermions. Whereas, in the 3-3-1 model with right-handed neutrinos the mechanism can work up to a very high scale such as the GUT's. In our framework, a combination of type I and II seesaws is always in the cooperation.

We have also argued that due to anomaly cancelation the 3-3-1 models may naturally permit of flavor symmetries such as S_4 and S_3 which has been taken into account since they possess $\underline{2}$ representations responsible for the 3-3-1 quark sector and $\mu - \tau$ symmetry as known. In addition, the 3-3-1 models can work only with three families as the flavor symmetries do. In the standard model, the families are in replication, thus naturally to put all in $\underline{3}$ which is $\underline{1} \oplus \underline{2}$ under S_3 . By this indication, we have put the first family in $\underline{1}$ and the last two in $\underline{2}$ to realize successful mass spectra and mixings for leptons and quarks.

We have introduced a new charge $U(1)_{\mathcal{L}}$ responsible for lepton number and lepton parity. The two 3-3-1 models as given are already in difference due to \mathcal{L} -charge embedding. In the 3-3-1 model with neutral fermions, the N_{RS} have vanishing lepton number and the lepton parity being in operation to realize the TeV seesaw mechanism in similarity to the scenario previously proposed [25]. In the 3-3-1 model with right-handed neutrinos, the lepton parity cannot work which realizes the popular seesaw mechanism. The scalar sector for the two models is also in difference. We have briefly discussed that if $U(1)_{\mathcal{L}}$ is violated via the scalar potential as given in Apps. C, the seesaw contributions are generated to be naturally small, responsible for the observed neutrino masses. By this reason the tree level exotic and ordinary quark mixing and flavor changing neutral current are also strongly suppressed. All those are in similarity to [10, 11, 23, 25].

We have shown that the realistic neutrino mixing can be obtained if the two directions for breaking $S_3 \rightarrow Z_2$ and $S_3 \rightarrow Z_3$ (or $S_3 \rightarrow Z_2 \rightarrow \{\text{Identity}\}$) simultaneously take place and equivalent in size, i.e. the contributions due to ρ (or ϕ') and s are comparable. If the s' which is also responsible for $Z_2 \rightarrow \{\text{Identity}\}$ breaking is not introduced, the models can fit the old data with $\theta_{13} = 0$. Otherwise, if it is presented as a small perturbation in contributing to the mass spectrum the new data under the light of the new observations can be naturally recognized.

Acknowledgments

This work was supported in part by the National Foundation for Science and Technology Development of Vietnam (NAFOSTED).

Appendix A: S_3 group and Clebsch-Gordan coefficients

S_3 is the permutation group of three objects, having six elements divided into three conjugacy classes [19]. We denote $\underline{1}$, $\underline{1}'$, and $\underline{2}$ as its three irreducible representations, and n , h as the order of class and the order of elements within each class, respectively. The character table is given by

Class	n	h	$\chi_{\underline{1}}$	$\chi_{\underline{1}'}$	$\chi_{\underline{2}}$
C_1	1	1	1	1	2
C_2	2	3	1	1	-1
C_3	3	2	1	-1	0

We will work in a basis in which the representation $\underline{2}$ is complex (See, for example, Ma in [19]). The decomposition rules can be obtained as

$$\begin{aligned}
\underline{1} \otimes \underline{1} &= \underline{1}(11), & \underline{1}' \otimes \underline{1}' &= \underline{1}(11), & \underline{1} \otimes \underline{1}' &= \underline{1}'(11), \\
\underline{1} \otimes \underline{2} &= \underline{2}(11, 12), & \underline{1}' \otimes \underline{2} &= \underline{2}(11, -12), \\
\underline{2} \otimes \underline{2} &= \underline{1}(12 + 21) \oplus \underline{1}'(12 - 21) \oplus \underline{2}(22, 11),
\end{aligned}$$

where for the terms in parentheses the first and second factor indicate to the multiplet components of the first and second representations, respectively. The conjugation rules are given by

$$\underline{2}^*(1^*, 2^*) = \underline{2}(2^*, 1^*), \quad \underline{1}^*(1^*) = \underline{1}(1^*), \quad \underline{1}'^*(1^*) = \underline{1}'(1^*).$$

Appendix B: Lepton number

For convenience, we also list the lepton number (L) of particles for the two models as mentioned.

1. The 3-3-1 model with neutral fermions

Particle	L
$u, d, N_R, W, Z, \phi_1^+, \phi_2^0, \phi_1'^+, \phi_2'^0, \eta_1^0, \eta_2^-, \eta_1'^0, \eta_2'^-, \chi_3^0, s_{33}^0, s_{33}'^0$	0
$\nu_L^*, l^*, U, D^*, X^{0*}, Y^+, \phi_3^+, \phi_3'^+, \eta_3^0, \eta_3'^0, \chi_1^{0*}, \chi_2^+, s_{13}^0, s_{23}^+, s_{13}'^0, s_{23}'^+, \rho_1^+, \rho_2^0$	-1
$s_{11}^0, s_{12}^+, s_{22}^{++}, s_{11}'^0, s_{12}'^+, s_{22}'^{++}, \rho_3^+$	-2

2. The 3-3-1 model with right-handed neutrinos

Particle	L
ν_L, ν_R, l	1
$u, d, W, Z, \phi_1^+, \phi_2^0, \phi_1'^+, \phi_2'^0, \eta_1^0, \eta_2^-, \eta_1'^0, \eta_2'^-, \chi_3^0, s_{13}^0, s_{23}^+, s_{13}'^0, s_{23}'^+$	0
$U, D^*, X^{0*}, Y^+, \phi_3^+, \phi_3'^+, \eta_3^0, \eta_3'^0, \chi_1^{0*}, \chi_2^+, s_{33}^{0*}, s_{33}'^{0*}, s_{11}^0, s_{12}^+, s_{22}^{++}, s_{11}'^0, s_{12}'^+, s_{22}'^{++}$	-2

Appendix C: Scalar potential

To be complete, we write the scalar potentials of both the models mentioned. It is also noted that $(\text{Tr}A)(\text{Tr}B) = \text{Tr}(A\text{Tr}B)$ and $V(X \rightarrow X', Y \rightarrow Y', \dots) \equiv V(X, Y, \dots)|_{X=X', Y=Y', \dots}$

1. The 3-3-1 model with neutral fermions

The general potential invariant under any group takes the form:

$$V_{\text{total}} = V_{\text{tri}} + V_{\text{sext}} + V_{\text{tri-sext}}, \quad (\text{C1})$$

where V_{tri} comes from only contributions of $\text{SU}(3)_L$ triplets given as a sum of:

$$V(\chi) = \mu_\chi^2 \chi^\dagger \chi + \lambda^\chi (\chi^\dagger \chi)^2, \quad (\text{C2})$$

$$V(\phi) = V(\chi \rightarrow \phi), \quad V(\phi') = V(\chi \rightarrow \phi'), \quad V(\eta) = V(\chi \rightarrow \eta), \quad (\text{C3})$$

$$V(\eta') = V(\chi \rightarrow \eta'), \quad V(\rho) = V(\chi \rightarrow \rho), \quad (\text{C4})$$

$$V(\phi, \chi) = \lambda_1^{\phi\chi} (\phi^\dagger \phi) (\chi^\dagger \chi) + \lambda_2^{\phi\chi} (\phi^\dagger \chi) (\chi^\dagger \phi), \quad (\text{C5})$$

$$V(\phi, \phi') = V(\phi, \chi \rightarrow \phi') + \lambda_3^{\phi\phi'} (\phi^\dagger \phi') (\phi^\dagger \phi') + \lambda_4^{\phi\phi'} (\phi'^\dagger \phi) (\phi'^\dagger \phi), \quad (\text{C6})$$

$$V(\phi, \eta) = V(\phi, \chi \rightarrow \eta), \quad V(\phi, \eta') = V(\phi, \chi \rightarrow \eta'), \quad (\text{C7})$$

$$V(\phi, \rho) = V(\phi, \chi \rightarrow \rho), \quad V(\phi', \chi) = V(\phi \rightarrow \phi', \chi), \quad (\text{C8})$$

$$V(\phi', \eta) = V(\phi \rightarrow \phi', \chi \rightarrow \eta), \quad V(\phi', \eta') = V(\phi \rightarrow \phi', \chi \rightarrow \eta'), \quad (\text{C9})$$

$$V(\phi', \rho) = V(\phi \rightarrow \phi', \chi \rightarrow \rho), \quad V(\chi, \eta) = V(\phi \rightarrow \chi, \chi \rightarrow \eta), \quad (\text{C10})$$

$$V(\chi, \eta') = V(\phi \rightarrow \chi, \chi \rightarrow \eta'), \quad V(\chi, \rho) = V(\phi \rightarrow \chi, \chi \rightarrow \rho), \quad (\text{C11})$$

$$V(\eta, \eta') = V(\phi \rightarrow \eta, \chi \rightarrow \eta') + \lambda_3^{\eta\eta'} (\eta^\dagger \eta') (\eta^\dagger \eta') + \lambda_4^{\eta\eta'} (\eta'^\dagger \eta) (\eta'^\dagger \eta), \quad (\text{C12})$$

$$V(\eta, \rho) = V(\phi \rightarrow \eta, \chi \rightarrow \rho), \quad V(\eta', \rho) = V(\phi \rightarrow \eta', \chi \rightarrow \rho), \quad (\text{C13})$$

$$\begin{aligned} V_{\chi\phi\phi'\eta\eta'\rho} = & \mu_1 \chi \phi \eta + \mu'_1 \chi \phi' \eta' + \lambda_1^1 (\phi^\dagger \phi') (\eta^\dagger \eta') + \lambda_1^2 (\phi^\dagger \phi') (\eta'^\dagger \eta) + \lambda_1^3 (\phi^\dagger \eta) (\eta'^\dagger \phi') \\ & + \lambda_1^4 (\phi^\dagger \eta') (\eta^\dagger \phi') + \lambda_1^5 (\phi^\dagger \rho) (\eta'^\dagger \chi) + \lambda_1^6 (\phi'^\dagger \rho) (\eta^\dagger \chi) + \lambda_1^7 (\eta'^\dagger \rho) (\phi^\dagger \chi) \\ & + \lambda_1^8 (\eta^\dagger \rho) (\phi'^\dagger \chi) + h.c. \end{aligned} \quad (\text{C14})$$

The V_{sext} is summed from only antisextet contributions:

$$\begin{aligned} V(s) = & \mu_s^2 \text{Tr}(s^\dagger s) + \lambda_1^s \text{Tr}[(s^\dagger s)_{\underline{1}}(s^\dagger s)_{\underline{1}}] + \lambda_2^s \text{Tr}[(s^\dagger s)_{\underline{1}'}(s^\dagger s)_{\underline{1}'}] + \lambda_3^s \text{Tr}[(s^\dagger s)_{\underline{2}}(s^\dagger s)_{\underline{2}}] \\ & + \lambda_4^s \text{Tr}(s^\dagger s)_{\underline{1}} \text{Tr}(s^\dagger s)_{\underline{1}} + \lambda_5^s \text{Tr}(s^\dagger s)_{\underline{1}'} \text{Tr}(s^\dagger s)_{\underline{1}'} + \lambda_6^s \text{Tr}(s^\dagger s)_{\underline{2}} \text{Tr}(s^\dagger s)_{\underline{2}}, \end{aligned} \quad (\text{C15})$$

$$V(s') = V(s \rightarrow s'), \quad (\text{C16})$$

$$\begin{aligned} V(s, s') = & \mu_{ss'}^2 \text{Tr}(s^\dagger s') + \lambda_1^{ss'} \text{Tr}[(s^\dagger s)_{\underline{1}}(s'^\dagger s')_{\underline{1}}] + \lambda_2^{ss'} \text{Tr}[(s^\dagger s)_{\underline{1}'}(s'^\dagger s')_{\underline{1}'}] + \lambda_3^{ss'} \text{Tr}[(s^\dagger s)_{\underline{2}}(s'^\dagger s')_{\underline{2}}] \\ & + \lambda_4^{ss'} \text{Tr}(s^\dagger s)_{\underline{1}} \text{Tr}(s'^\dagger s')_{\underline{1}} + \lambda_5^{ss'} \text{Tr}(s^\dagger s)_{\underline{1}'} \text{Tr}(s'^\dagger s')_{\underline{1}'} + \lambda_6^{ss'} \text{Tr}(s^\dagger s)_{\underline{2}} \text{Tr}(s'^\dagger s')_{\underline{2}} \\ & + \lambda_7^{ss'} \text{Tr}[(s^\dagger s')_{\underline{1}}(s'^\dagger s)_{\underline{1}}] + \lambda_8^{ss'} \text{Tr}[(s^\dagger s')_{\underline{1}'}(s'^\dagger s)_{\underline{1}'}] + \lambda_9^{ss'} \text{Tr}[(s^\dagger s')_{\underline{2}}(s'^\dagger s)_{\underline{2}}] \\ & + \lambda_{10}^{ss'} \text{Tr}(s^\dagger s')_{\underline{1}} \text{Tr}(s'^\dagger s)_{\underline{1}} + \lambda_{11}^{ss'} \text{Tr}(s^\dagger s')_{\underline{1}'} \text{Tr}(s'^\dagger s)_{\underline{1}'} + \lambda_{12}^{ss'} \text{Tr}(s^\dagger s')_{\underline{2}} \text{Tr}(s'^\dagger s)_{\underline{2}} \\ & + \lambda_{13}^{ss'} \text{Tr}[(s^\dagger s')_{\underline{1}}(s^\dagger s')_{\underline{1}}] + \lambda_{14}^{ss'} \text{Tr}[(s^\dagger s')_{\underline{1}'}(s^\dagger s')_{\underline{1}'}] + \lambda_{15}^{ss'} \text{Tr}[(s^\dagger s')_{\underline{2}}(s^\dagger s')_{\underline{2}}] \\ & + \lambda_{16}^{ss'} \text{Tr}(s^\dagger s')_{\underline{1}} \text{Tr}(s^\dagger s')_{\underline{1}} + \lambda_{17}^{ss'} \text{Tr}(s^\dagger s')_{\underline{1}'} \text{Tr}(s^\dagger s')_{\underline{1}'} + \lambda_{18}^{ss'} \text{Tr}(s^\dagger s')_{\underline{2}} \text{Tr}(s^\dagger s')_{\underline{2}} \\ & + \lambda_{19}^{ss'} \text{Tr}[(s^\dagger s)_{\underline{1}}(s^\dagger s')_{\underline{1}}] + \lambda_{20}^{ss'} \text{Tr}[(s^\dagger s)_{\underline{1}'}(s^\dagger s')_{\underline{1}'}] + \lambda_{21}^{ss'} \text{Tr}[(s^\dagger s)_{\underline{2}}(s^\dagger s')_{\underline{2}}] \\ & + \lambda_{22}^{ss'} \text{Tr}(s^\dagger s)_{\underline{1}} \text{Tr}(s^\dagger s')_{\underline{1}} + \lambda_{23}^{ss'} \text{Tr}(s^\dagger s)_{\underline{1}'} \text{Tr}(s^\dagger s')_{\underline{1}'} + \lambda_{24}^{ss'} \text{Tr}(s^\dagger s)_{\underline{2}} \text{Tr}(s^\dagger s')_{\underline{2}} \\ & + \lambda_{25}^{ss'} \text{Tr}[(s'^\dagger s')_{\underline{1}}(s^\dagger s')_{\underline{1}}] + \lambda_{26}^{ss'} \text{Tr}[(s'^\dagger s')_{\underline{1}'}(s^\dagger s')_{\underline{1}'}] + \lambda_{27}^{ss'} \text{Tr}[(s'^\dagger s')_{\underline{2}}(s^\dagger s')_{\underline{2}}] \\ & + \lambda_{28}^{ss'} \text{Tr}(s'^\dagger s')_{\underline{1}} \text{Tr}(s^\dagger s')_{\underline{1}} + \lambda_{29}^{ss'} \text{Tr}(s'^\dagger s')_{\underline{1}'} \text{Tr}(s^\dagger s')_{\underline{1}'} + \lambda_{30}^{ss'} \text{Tr}(s'^\dagger s')_{\underline{2}} \text{Tr}(s^\dagger s')_{\underline{2}} \\ & + h.c. \end{aligned} \quad (\text{C17})$$

The $V_{\text{tri-sext}}$ is given as a sum of all the terms connecting both the sectors:

$$V(\phi, s) = \lambda_1^{\phi s} (\phi^\dagger \phi) \text{Tr}(s^\dagger s)_{\underline{1}} + \lambda_2^{\phi s} [(\phi^\dagger s^\dagger)(s\phi)]_{\underline{1}}, \quad (\text{C18})$$

$$V(\phi', s) = V(\phi \rightarrow \phi', s), \quad V(\chi, s) = V(\phi \rightarrow \chi, s), \quad V(\eta, s) = V(\phi \rightarrow \eta, s), \quad (\text{C19})$$

$$V(\eta', s) = V(\phi \rightarrow \eta', s), \quad V(\rho, s) = V(\phi \rightarrow \rho, s) + \{\lambda_3^{\rho s} \rho [(\rho s^\dagger) s^\dagger]_{\underline{1}'} + h.c.\}, \quad (\text{C20})$$

$$V(\phi, s') = V(\phi, s \rightarrow s'), \quad V(\phi', s') = V(\phi \rightarrow \phi', s \rightarrow s'), \quad (\text{C21})$$

$$V(\chi, s') = V(\phi \rightarrow \chi, s \rightarrow s'), \quad V(\eta, s') = V(\phi \rightarrow \eta, s \rightarrow s'), \quad (\text{C22})$$

$$V(\eta', s') = V(\phi \rightarrow \eta', s \rightarrow s'), \quad (C23)$$

$$V(\rho, s') = V(\phi \rightarrow \rho, s \rightarrow s') + \{\lambda_3^{\rho s'} \rho[(\rho s^\dagger) s'^\dagger]_{\underline{1}'} + h.c.\}, \quad (C24)$$

$$V(\phi, s, s') = \lambda_1^{\phi s s'} (\phi^\dagger \phi) \text{Tr}(s^\dagger s')_{\underline{1}} + \lambda_2^{\phi s s'} [(\phi^\dagger s^\dagger)(s' \phi)]_{\underline{1}} + h.c., \quad (C25)$$

$$V(\phi', s, s') = V(\phi \rightarrow \phi', s, s'), \quad V(\chi, s, s') = V(\phi \rightarrow \chi, s, s'), \quad (C26)$$

$$V(\eta, s, s') = V(\phi \rightarrow \eta, s, s'), \quad V(\eta', s, s') = V(\phi \rightarrow \eta', s, s'), \quad (C27)$$

$$V(\rho, s, s') = V(\phi \rightarrow \rho, s, s') + \{\lambda_3^{\rho s s'} \rho[(\rho s^\dagger) s'^\dagger]_{\underline{1}'} + \lambda_4^{\rho s s'} \rho[(\rho s'^\dagger) s^\dagger]_{\underline{1}'} + h.c.\}, \quad (C28)$$

$$\begin{aligned} V_{ss' \chi \phi \phi' \eta \eta' \rho} = & (\lambda_1 \phi^\dagger \phi' + \lambda_2 \eta^\dagger \eta') \text{Tr}(s^\dagger s)_{\underline{1}'} + \lambda_3 [(\phi^\dagger s^\dagger)(s \phi')]_{\underline{1}} + \lambda_4 [(\eta^\dagger s^\dagger)(s \eta')]_{\underline{1}} \\ & + (\lambda_5 \phi^\dagger \phi' + \lambda_6 \eta^\dagger \eta') \text{Tr}(s'^\dagger s')_{\underline{1}'} + \lambda_7 [(\phi^\dagger s'^\dagger)(s' \phi')]_{\underline{1}} + \lambda_8 [(\eta^\dagger s'^\dagger)(s' \eta')]_{\underline{1}} \\ & + (\lambda_9 \phi^\dagger \phi' + \lambda_{10} \eta^\dagger \eta') \text{Tr}(s^\dagger s')_{\underline{1}'} + \lambda_{11} [(\phi^\dagger s^\dagger)(s' \phi')]_{\underline{1}} + \lambda_{12} [(\eta^\dagger s^\dagger)(s' \eta')]_{\underline{1}} \\ & + (\lambda_{13} \phi^\dagger \phi' + \lambda_{14} \eta^\dagger \eta') \text{Tr}(s'^\dagger s)_{\underline{1}'} + \lambda_{15} [(\phi^\dagger s'^\dagger)(s \phi')]_{\underline{1}} + \lambda_{16} [(\eta^\dagger s'^\dagger)(s \eta')]_{\underline{1}} \\ & + h.c. \end{aligned} \quad (C29)$$

To provide the Majorana masses for the neutrinos, the lepton number must be broken. This can be achieved via the scalar potential violating $U(1)_{\mathcal{L}}$, however the other symmetries should be conserved. The \mathcal{L} violating potential is given as

$$\begin{aligned} \bar{V} = & \bar{\mu} \chi \rho \eta' + [\bar{\lambda}_1 \phi^\dagger \phi + \bar{\lambda}_2 \phi'^\dagger \phi' + \bar{\lambda}_3 \chi^\dagger \chi + \bar{\lambda}_4 \eta^\dagger \eta + \bar{\lambda}_5 \eta'^\dagger \eta' + \bar{\lambda}_6 \rho^\dagger \rho + \bar{\lambda}_7 \eta^\dagger \chi + \bar{\lambda}_8 \rho^\dagger \phi' \\ & + \bar{\lambda}_9 \text{Tr}(s^\dagger s)_{\underline{1}} + \bar{\lambda}_{10} \text{Tr}(s'^\dagger s')_{\underline{1}} + \bar{\lambda}_{11} \text{Tr}(s^\dagger s')_{\underline{1}} + \bar{\lambda}_{12} \text{Tr}(s'^\dagger s)_{\underline{1}}] (\eta^\dagger \chi) + [\bar{\lambda}_{13} \phi^\dagger \phi' + \bar{\lambda}_{14} \phi'^\dagger \phi \\ & + \bar{\lambda}_{15} \eta^\dagger \eta' + \bar{\lambda}_{16} \eta'^\dagger \eta + \bar{\lambda}_{17} \eta^\dagger \chi + \bar{\lambda}_{18} \rho^\dagger \phi + \bar{\lambda}_{19} \text{Tr}(s^\dagger s)_{\underline{1}'} + \bar{\lambda}_{20} \text{Tr}(s'^\dagger s')_{\underline{1}'} + \bar{\lambda}_{21} \text{Tr}(s^\dagger s')_{\underline{1}'} \\ & + \bar{\lambda}_{22} \text{Tr}(s'^\dagger s)_{\underline{1}'}] (\eta'^\dagger \chi) + [\bar{\lambda}_{23} \eta^\dagger \phi + \bar{\lambda}_{24} \eta'^\dagger \phi' + \bar{\lambda}_{25} \eta'^\dagger \rho] (\phi^\dagger \chi) + [\bar{\lambda}_{26} \eta^\dagger \phi' + \bar{\lambda}_{27} \eta'^\dagger \phi + \bar{\lambda}_{28} \chi^\dagger \rho \\ & + \bar{\lambda}_{29} \eta^\dagger \rho] (\phi'^\dagger \chi) + [\bar{\lambda}_{30} \phi^\dagger \phi' + \bar{\lambda}_{31} \phi'^\dagger \phi + \bar{\lambda}_{32} \eta^\dagger \eta' + \bar{\lambda}_{33} \eta'^\dagger \eta + \bar{\lambda}_{34} \text{Tr}(s^\dagger s)_{\underline{1}'} + \bar{\lambda}_{35} \text{Tr}(s'^\dagger s')_{\underline{1}'} \\ & + \bar{\lambda}_{36} \text{Tr}(s^\dagger s')_{\underline{1}'} + \bar{\lambda}_{37} \text{Tr}(s'^\dagger s)_{\underline{1}'}] (\phi^\dagger \rho) + [\bar{\lambda}_{38} \phi^\dagger \phi + \bar{\lambda}_{39} \phi'^\dagger \phi' + \bar{\lambda}_{40} \chi^\dagger \chi + \bar{\lambda}_{41} \eta^\dagger \eta + \bar{\lambda}_{42} \eta'^\dagger \eta' \\ & + \bar{\lambda}_{43} \text{Tr}(s^\dagger s)_{\underline{1}} + \bar{\lambda}_{44} \text{Tr}(s'^\dagger s')_{\underline{1}} + \bar{\lambda}_{45} \text{Tr}(s^\dagger s')_{\underline{1}} + \bar{\lambda}_{46} \text{Tr}(s'^\dagger s)_{\underline{1}}] (\phi'^\dagger \rho) + [\bar{\lambda}_{47} \phi^\dagger \eta' + \bar{\lambda}_{48} \phi'^\dagger \eta] (\chi^\dagger \rho) \\ & + [\bar{\lambda}_{49} \phi^\dagger \eta' + \bar{\lambda}_{50} \phi'^\dagger \eta] (\eta^\dagger \rho) + [\bar{\lambda}_{51} \phi^\dagger \eta + \bar{\lambda}_{52} \phi'^\dagger \eta'] (\eta'^\dagger \rho) + \bar{\lambda}_{53} [(\eta^\dagger s^\dagger)(s \chi)]_{\underline{1}} + \bar{\lambda}_{54} [(\eta^\dagger s'^\dagger)(s' \chi)]_{\underline{1}} \\ & + \bar{\lambda}_{55} [(\eta^\dagger s^\dagger)(s' \chi)]_{\underline{1}} + \bar{\lambda}_{56} [(\eta^\dagger s'^\dagger)(s \chi)]_{\underline{1}} + \bar{\lambda}_{57} [(\eta'^\dagger s^\dagger)(s \chi)]_{\underline{1}} + \bar{\lambda}_{58} [(\eta'^\dagger s'^\dagger)(s' \chi)]_{\underline{1}} \\ & + \bar{\lambda}_{59} [(\eta'^\dagger s^\dagger)(s' \chi)]_{\underline{1}} + \bar{\lambda}_{60} [(\eta'^\dagger s'^\dagger)(s \chi)]_{\underline{1}} + \bar{\lambda}_{61} [(\rho^\dagger s^\dagger)(s \phi)]_{\underline{1}} + \bar{\lambda}_{62} [(\rho^\dagger s'^\dagger)(s' \phi)]_{\underline{1}} \\ & + \bar{\lambda}_{63} [(\rho^\dagger s^\dagger)(s' \phi)]_{\underline{1}} + \bar{\lambda}_{64} [(\rho^\dagger s'^\dagger)(s \phi)]_{\underline{1}} + \bar{\lambda}_{65} [(\rho^\dagger s^\dagger)(s \phi')]_{\underline{1}} + \bar{\lambda}_{66} [(\rho^\dagger s'^\dagger)(s' \phi')]_{\underline{1}} \\ & + \bar{\lambda}_{67} [(\rho^\dagger s^\dagger)(s' \phi')]_{\underline{1}} + \bar{\lambda}_{68} [(\rho^\dagger s'^\dagger)(s \phi')]_{\underline{1}} + \bar{\lambda}_{69} \phi [(\phi s^\dagger) s^\dagger]_{\underline{1}} + \bar{\lambda}_{70} \phi [(\phi s'^\dagger) s'^\dagger]_{\underline{1}} + \bar{\lambda}_{71} \phi [(\phi s^\dagger) s'^\dagger]_{\underline{1}} \\ & + \bar{\lambda}_{72} \phi [(\phi s'^\dagger) s^\dagger]_{\underline{1}} + \bar{\lambda}_{73} \phi' [(\phi' s^\dagger) s^\dagger]_{\underline{1}'} + \bar{\lambda}_{74} \phi' [(\phi' s'^\dagger) s'^\dagger]_{\underline{1}'} + \bar{\lambda}_{75} \phi' [(\phi' s^\dagger) s'^\dagger]_{\underline{1}'} + \bar{\lambda}_{76} \phi' [(\phi' s'^\dagger) s^\dagger]_{\underline{1}'} \\ & + \bar{\lambda}_{77} \phi [(\phi' s^\dagger) s^\dagger]_{\underline{1}} + \bar{\lambda}_{78} \phi [(\phi' s'^\dagger) s'^\dagger]_{\underline{1}} + \bar{\lambda}_{79} \phi [(\phi' s^\dagger) s'^\dagger]_{\underline{1}} + \bar{\lambda}_{80} \phi [(\phi' s'^\dagger) s^\dagger]_{\underline{1}} + \bar{\lambda}_{81} \phi' [(\phi s^\dagger) s^\dagger]_{\underline{1}'} \\ & + \bar{\lambda}_{82} \phi' [(\phi s'^\dagger) s'^\dagger]_{\underline{1}'} + \bar{\lambda}_{83} \phi' [(\phi s^\dagger) s'^\dagger]_{\underline{1}'} + \bar{\lambda}_{84} \phi' [(\phi s'^\dagger) s^\dagger]_{\underline{1}'} + h.c. \end{aligned} \quad (C30)$$

2. The 3-3-1 model with right-handed neutrinos

The general potential is given as that of the 3-3-1 model with neutral fermions but all the terms relevant to the ρ scalar should be suppressed. In addition, $V_{\text{tri-sext}}$ contains extra terms as follows:

$$\begin{aligned}
& \lambda_{17}\phi[(\phi s^\dagger)s^\dagger]_{\underline{1}} + \lambda_{18}\phi[(\phi s'^\dagger)s'^\dagger]_{\underline{1}} + \lambda_{19}\phi[(\phi s^\dagger)s'^\dagger]_{\underline{1}} + \lambda_{20}\phi[(\phi s'^\dagger)s^\dagger]_{\underline{1}} \\
& + \lambda_{21}\phi'[(\phi' s^\dagger)s^\dagger]_{\underline{1}'} + \lambda_{22}\phi'[(\phi' s'^\dagger)s'^\dagger]_{\underline{1}'} + \lambda_{23}\phi'[(\phi' s^\dagger)s'^\dagger]_{\underline{1}'} + \lambda_{24}\phi'[(\phi' s'^\dagger)s^\dagger]_{\underline{1}'} \\
& + \lambda_{25}\phi[(\phi' s^\dagger)s^\dagger]_{\underline{1}} + \lambda_{26}\phi[(\phi' s'^\dagger)s'^\dagger]_{\underline{1}} + \lambda_{27}\phi[(\phi' s^\dagger)s'^\dagger]_{\underline{1}} + \lambda_{28}\phi[(\phi' s'^\dagger)s^\dagger]_{\underline{1}} \\
& + \lambda_{29}\phi'[(\phi s^\dagger)s^\dagger]_{\underline{1}'} + \lambda_{30}\phi'[(\phi s'^\dagger)s'^\dagger]_{\underline{1}'} + \lambda_{31}\phi'[(\phi s^\dagger)s'^\dagger]_{\underline{1}'} + \lambda_{32}\phi'[(\phi s'^\dagger)s^\dagger]_{\underline{1}'} \\
& + h.c.
\end{aligned} \tag{C31}$$

It is noted that in the 3-3-1 model with neutral fermions the similar couplings appear, however, in the \mathcal{L} violating potential (C30). Therefore it will disappear in the one mentioned below.

The \mathcal{L} violating potential for this model is similar to that of the 3-3-1 model with neutral fermions (C30), however all the interactions therein that have appeared in (C31) must be removed.

-
- [1] K. Nakamura *et al.* [Particle Data Group], J. Phys. G **37**, 075021 (2010).
 - [2] P. Minkowski, Phys. Lett. B **67**, 421 (1977); M. Gell-Mann, P. Ramond and R. Slansky, *Complex spinors and unified theories*, in *Supergravity*, edited by P. van Nieuwenhuizen and D. Z. Freedman (North Holland, Amsterdam, 1979), p. 315; T. Yanagida, in *Proceedings of the Workshop on the Unified Theory and the Baryon Number in the Universe*, edited by O. Sawada and A. Sugamoto (KEK, Tsukuba, Japan, 1979), p. 95; S. L. Glashow, *The future of elementary particle physics*, in *Proceedings of the 1979 Cargèse Summer Institute on Quarks and Leptons*, edited by M. Lévy et al. (Plenum Press, New York, 1980), pp. 687-713; R. N. Mohapatra and G. Senjanović, Phys. Rev. Lett. **44**, 912 (1980).
 - [3] M. Fukugita and T. Yanagida, Phys. Lett. B **174**, 45 (1986).
 - [4] J. C. Pati and A. Salam, Phys. Rev. D **10**, 275 (1974); R. N. Mohapatra and J. C. Pati, Phys. Rev. D **11**, 566, 2558 (1975); G. Senjanović and R. N. Mohapatra, Phys. Rev. D **12**, 1502 (1975).
 - [5] H. Georgi, in *Particles and Fields*, edited by C. E. Carlson (A.I.P., New York, 1975); H. Fritzsch and P. Minkowski, Ann. Phys. **93**, 193 (1975).
 - [6] M. Singer, J. W. F. Valle and J. Schechter, Phys. Rev. D **22**, 738 (1980); J. C. Montero, F. Pisano and V. Pleitez, Phys. Rev. D **47**, 2918 (1993); R. Foot, H. N. Long and Tuan A. Tran, Phys. Rev. D **50**, 34(R) (1994); H. N. Long, Phys. Rev. D **53**, 437 (1996); **54**, 4691 (1996).
 - [7] F. Pisano and V. Pleitez, Phys. Rev. D **46**, 410 (1992); P. H. Frampton, Phys. Rev. Lett. **69**, 2889 (1992); R. Foot, O. F. Hernandez, F. Pisano and V. Pleitez, Phys. Rev. D **47**, 4158 (1993).

- [8] W. A. Ponce, Y. Giraldo and L. A. Sanchez, Phys. Rev. D **67**, 075001 (2003); P. V. Dong, H. N. Long, D. T. Nhung and D. V. Soa, Phys. Rev. D **73**, 035004 (2006); P. V. Dong and H. N. Long, Adv. High Energy Phys. **2008**, 739492 (2008).
- [9] A. Doff and F. Pisano, Mod. Phys. Lett. A **15**, 1471 (2000); P. V. Dong and H. N. Long, Int. J. Mod. Phys. A **21** 6677, (2006).
- [10] P. V. Dong, L. T. Hue, H. N. Long and D. V. Soa, Phys. Rev. D **81**, 053004 (2010).
- [11] P. V. Dong, H. N. Long, D. V. Soa, and V. V. Vien, Eur. Phys. J. C **71**, 1544 (2011).
- [12] P. F. Harrison, D. H. Perkins and W. G. Scott, Phys. Lett. B **530**, 167 (2002); Z. Z. Xing, Phys. Lett. B **533**, 85 (2002); X. G. He and A. Zee, Phys. Lett. B **560**, 87 (2003); Phys. Rev. D **68**, 037302 (2003).
- [13] E. Ma and G. Rajasekaran, Phys. Rev. D **64**, 113012 (2001); K. S. Babu, E. Ma and J. W. F. Valle, Phys. Lett. B **552**, 207 (2003); G. Altarelli and F. Feruglio, Nucl. Phys. B **720**, 64 (2005); E. Ma, Phys. Rev. D **73**, 057304 (2006); X. G. He, Y. Y. Keum and R. R. Volkas, JHEP **0604**, 039 (2006); S. Morisi, M. Picariello, and E. Torrente-Lujan, Phys. Rev. D **75**, 075015 (2007); C. S. Lam, Phys. Lett. B **656**, 193 (2007); F. Bazzocchi, S. Kaneko and S. Morisi, JHEP **0803**, 063 (2008); A. Blum, C. Hagedorn, and M. Lindner, Phys. Rev. D **77**, 076004 (2008); F. Bazzocchi, M. Frigerio, and S. Morisi, Phys. Rev. D **78**, 116018 (2008); G. Altarelli, F. Feruglio and C. Hagedorn, JHEP **0803**, 052 (2008); M. Hirsch, S. Morisi and J. W. F. Valle, Phys. Rev. D **78**, 093007 (2008); E. Ma, Phys. Lett. B **671**, 366 (2009); G. Altarelli and D. Meloni, J. Phys. G **36**, 085005 (2009); Y. Lin, Nucl. Phys. B **813**, 91 (2009); Y. H. Ahn and C. S. Chen, Phys. Rev. D **81**, 105013 (2010); J. Barry and W. Rodejohanny, Phys. Rev. D **81**, 093002 (2010); 119901(E) (2010); G. J. Ding and D. Meloni, Nucl. Phys. B **855**, 21 (2012).
- [14] R. N. Mohapatra, M. K. Parida, G. Rajasekaran, Phys. Rev. D **69**, 053007 (2004); C. Hagedorn, M. Lindner, and R. N. Mohapatra, JHEP **0606**, 042 (2006); E. Ma, Phys. Lett. B **632**, 352 (2006); H. Zhang, Phys. Lett. B **655**, 132 (2007); Y. Koide, JHEP **0708**, 086 (2007); H. Ishimori, Y. Shimizu, and M. Tanimoto, Prog. Theor. Phys. **121**, 769 (2009); F. Bazzocchi, L. Merlo, and S. Morisi, Nucl. Phys. B **816**, 204 (2009); F. Bazzocchi and S. Morisi, Phys. Rev. D **80** 096005 (2009); G. Altarelli and F. Feruglio, Rev. Mod. Phys. **82**, 2701 (2010); G. J. Ding, Nucl. Phys. B **827**, 82 (2010); Y. H. Ahn, S. K. Kang, C. S. Kim, and T. P. Nguyen, Phys. Rev. D **82**, 093005 (2010); Y. Daikoku and H. Okada, arXiv:1008.0914 [hep-ph]; H. Ishimori, Y. Shimizu, M. Tanimoto, and A. Watanabe, Phys. Rev. D **83** 033004 (2011); H. Ishimori and M. Tanimoto, Prog. Theor. Phys. **125**, 653 (2011); R. Z. Yang and H. Zhang, Phys. Lett. B **700** 316, (2011); S. Morisi and E. Peinado, Phys. Lett. B **701**, 451 (2011); S. Morisi, K. M. Patel, and E. Peinado, Phys. Rev. D **84**, 053002 (2011).
- [15] P. V. Dong and H. N. Long, Phys. Rev. D **77**, 057302 (2008).
- [16] K. Abe *et al.* [T2K Collaboration], Phys. Rev. Lett. **107**, 041801 (2011); P. Adamson *et al.* [MINOS Collaboration], arXiv:1108.0015 [hep-ex].
- [17] T. Schwetz, M. Tortola, and J. Valle, New J. Phys. **13**, 109401 (2011).
- [18] G. L. Fogli, E. Lisi, A. Marrone, A. Palazzo, and A. M. Rotunno, Phys. Rev. Lett. **101**, 141801 (2008); M. C. Gonzalez-Garcia, M. Maltoni, and J. Salvado, JHEP **1004**, 056 (2010); G. L. Fogli *et al.*, Phys.

- Rev. D **84** 053007 (2011); T. Schwetz, M. Tortola, and J. W. F. Valle, New J. Phys. **13**, 063004 (2011).
- [19] E. Ma, arXiv:hep-ph/0409075; H. Ishimori *et. al.*, Prog. Theor. Phys. Suppl. **183**, 1 (2010).
- [20] See, for an incomplete list, L. Wolfenstein, Phys. Rev. D **18**, 958 (1978); S. Pakvasa and H. Sugawara, Phys. Lett. B **73**, 61 (1978); **82**, 105 (1979); E. Durman and H. S. Tsao, Phys. Rev. D **20**, 1207 (1979); Y. Yamanaka, H. Sugawara, and S. Pakvasa, Phys. Rev. D **25**, 1895 (1982); K. Kang, J. E. Kim, and P. Ko, Z. Phys. C **72**, 671 (1996); H. Fritzsch and Z. Z. Xing, Phys. Lett. B **372**, 265 (1996); K. Kang, S. K. Kang, J. E. Kim, and P. Ko, Mod. Phys. Lett. A **12**, 1175 (1997); M. Fukugita, M. Tanimoto, and T. Yanagida, Phys. Rev. D **57**, 4429 (1998); H. Fritzsch and Z. Z. Xing, Phys. Lett. B **440**, 313 (1998); Y. Koide, Phys. Rev. D **60**, 077301 (1999); H. Fritzsch and Z. Z. Xing, Phys. Rev. D **61**, 073016 (2000); M. Tanimoto, Phys. Lett. B **483**, 417 (2000); G. C. Branco and J. I. Silva-Marcos, Phys. Lett. B **526**, 104 (2002); M. Fujii, K. Hamaguchi and T. Yanagida, Phys. Rev. D **65**, 115012 (2002); J. Kubo, A. Mondragon, M. Mondragon and E. Rodriguez-Jauregui, Prog. Theor. Phys. **109**, 795 (2003); **114**, 287(E) (2005); P. F. Harrison and W. G. Scott, Phys. Lett. B **557**, 76 (2003); S.-L. Chen, M. Frigerio, and E. Ma, Phys. Rev. D **70**, 073008 (2004); **70**, 079905(E) (2004); H. Fritzsch and Z. Z. Xing, Phys. Lett. B **598**, 237 (2004); F. Caravaglios and S. Morisi, arXiv:hep-ph/0503234; W. Grimus and L. Lavoura, JHEP **0508**, 013 (2005); R. N. Mohapatra, S. Nasri and H. B. Yu, Phys. Lett. B **639**, 318 (2006); R. Jora, S. Nasri and J. Schechter, Int. J. Mod. Phys. A **21**, 5875 (2006); J. E. Kim and J.-C. Park, JHEP **0605**, 017 (2006); Y. Koide, Eur. Phys. J. C **50**, 809 (2007); A. Mondragon, M. Mondragon, and E. Peinado, Phys. Rev. D **76**, 076003 (2007); AIP Conf. Proc. **1026**, 164 (2008); M. Picariello, Int. J. Mod. Phys. A **23**, 4435 (2008); C. Y. Chen and L. Wolfenstein, Phys. Rev. D **77**, 093009 (2008); R. Jora, J. Schechter and M. Naeem Shahid, Phys. Rev. D **80**, 093007 (2009); **82**, 079902(E) (2010); D. A. Dicus, S. F. Ge and W. W. Repko, Phys. Rev. D **82**, 033005 (2010); Z. Z. Xing, D. Yang and S. Zhou, Phys. Lett. B **690**, 304 (2010); R. Jora, J. Schechter and M. N. Shahid, Phys. Rev. D **82**, 053006 (2010); S. Dev, S. Gupta and R. R. Gautam, Phys. Lett. B **702**, 28 (2011); D. Meloni, S. Morisi and E. Peinado, J. Phys. G **38**, 015003 (2011); G. Bhattacharyya, P. Leser and H. Pas, Phys. Rev. D **83**, 011701(R) (2011); T. Kaneko and H. Sugawara, Phys. Lett. B **697**, 329 (2011); S. Zhou, Phys. Lett. B **704**, 291 (2011).
- [21] D. Chang and H. N. Long, Phys. Rev. D **73**, 053006 (2006).
- [22] E. Ma, Mod. Phys. Lett. A **25**, 2215 (2010).
- [23] See, for example, E. Ma and U. Sarkar, Phys. Rev. Lett. **80**, 5716 (1998).
- [24] R. N. Mohapatra and G. Senjanović, Phys. Rev. D **23**, 165 (1981); G. Lazarides, Q. Shafi and C. Wetterich, Nucl. Phys. B **181**, 287 (1981); J. Schechter and J. W. Valle, Phys. Rev. D **25**, 774 (1982).
- [25] E. Ma, Phys. Rev. Lett. **86**, 2502 (2001).
- [26] P. V. Dong, H. N. Long and D. V. Soa, Phys. Rev. D **75**, 073006 (2007).